Weak convergence analysis of the symmetrized Euler scheme for one dimensional SDEs with diffusion coefficient $|x|^\alpha$, $\alpha \in [\frac{1}{2}, 1)^*$

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Abstract

In this paper, we are interested in the time discrete approximation of $\mathbb{E}f(X_T)$ when X is the solution of a stochastic differential equation with a diffusion coefficient function of the form $|x|^\alpha$. We propose a symmetrized version of the Euler scheme, applied to X. The symmetrized version is very easy to simulate on a computer. For smooth functions f, we prove the Feynman Kac representation formula $u(t,x) = \mathbb{E}_{t,x}f(X_T)$, for u solving the associated Kolmogorov PDE and we obtain the upper-bounds on the spatial derivatives of u up to the order four. Then we show that the weak error of our symmetrized scheme is of order one, as for the classical Euler scheme.

Keywords. discretisation scheme; weak approximation MSC 65CXX, 60H35

1 Introduction

We consider $(X_t, t \geq 0)$, the $\mathbb{R}$-valued process solution to the following one-dimensional Itô stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma \int_0^t |X_s|^\alpha dW_s, \tag{1}$$

where x_0 and σ are given constants, $\sigma > 0$ and $(W_t, t \geq 0)$ is a one-dimensional Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $(\mathcal{F}_t, t \geq 0)$ the Brownian filtration. To ensure the existence of such process, we state the following hypotheses:

(H0) $\alpha \in [1/2, 1)$.

(H1) *The drift function b is such that $b(0) > 0$ and satisfies the Lipschitz condition*

$$|b(x) - b(y)| \leq K|x - y|,\ \forall\, (x,y) \in \mathbb{R}^2.$$

Under hypotheses (H0) and (H1), strong existence and uniqueness holds for equation (1). Moreover, when $x_0 \geq 0$ and $b(0) > 0$, the process $(X_t, t \geq 0)$ is valued in $[0, +\infty)$ (see e.g. [14]). Then (X) is the unique strong solution to

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma \int_0^t X_s^\alpha dW_s. \tag{2}$$

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Simulation schemes for Equation (1) are motivated by some applications in Finance: in [6], Cox, Ingersoll and Ross (CIR) proposed to model the dynamics of the short term interest rate as the solution of (1) with $\alpha = 1/2$ and $b(x) = a - bx$. Still to model the short term interest rate, Hull and White [13] proposed the following mean-reverting diffusion process

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)r_t^{\alpha}dW_t$$

with $0 \leq \alpha \leq 1$. More recently, the stochastic–$\alpha\beta\rho$ model or $SABR$–model have been proposed as a stochastic correlated volatility model for the asset price (see [12]):

$$\begin{aligned} dX_t &= \sigma_t X_t^{\beta} dW_t^1 \\ d\sigma_t &= \alpha\sigma_t dB_t \end{aligned}$$

where $B_t = \rho W_t^1 + \sqrt{(1-\rho^2)}W_t^2$, $\rho \in [-1, 1]$ and (W^1, W^2) is a 2d–Brownian motion.

CIR-like models arise also in fluid mechanics: in the stochastic Lagrangian modeling of turbulent flow, characteristic quantities like the instantaneous turbulent frequency (ω_t) are modeled by (see [9])

$$\begin{aligned} d\omega_t = -C_3\langle\omega_t\rangle\left(\omega_t - \langle\omega_t\rangle\right)dt - S(\langle\omega_t\rangle)\omega_t dt \\ + \sqrt{C_4\langle\omega_t\rangle^2\omega_t}dW_t \end{aligned}$$

where the ensemble average $\langle\omega_t\rangle$ denotes here the conditional expectation with respect to the position of the underlying portion of fluid and $S(\omega)$ is a given function.

In the examples above, the solution processes are all positive. In the practice, this could be an important feature of the model that simulation procedures have to preserve. By using the classical Euler scheme, one cannot define a positive approximation process. Similar situations occur when one consider discretisation scheme of a reflected stochastic differential equation. To maintain the approximation process in a given domain, an efficient strategy consists in symmetrizing the value obtained by the Euler scheme with respect to the boundary of the domain (see e.g. [4]). Here, our preoccupation is quite similar. We want to maintain the positive value of the approximation. In addition, we have to deal with a just locally Lipschitz diffusion coefficient.

In [7], Deelstra and Delbaen prove the strong convergence of the Euler scheme apply to $dX_t = \kappa(\gamma - X_t)dt + g(X_t)dW_t$ where $g : \mathbb{R} \to \mathbb{R}^+$ vanishes at zero and satisfies the Hölder condition $|g(x) - g(y)| \leq b\sqrt{|x-y|}$. The Euler scheme is applied to the modified equation $dX_t = \kappa(\gamma - X_t)dt + g(X_t \mathbb{1}_{\{X_t \geq 0\}})dW_t$. This corresponds to a projection scheme. For reflected SDEs, this procedure convergences weakly with a rate $\frac{1}{2}$ (see [5]). Moreover, the positivity of the simulated process is not guaranteed. In the particular case of the CIR processes, Alfonsi [1] proposes some implicit schemes, which admit analytical solutions, and derives from them a family of explicit schemes. He analyses their rate of convergence (in both strong and weak sense) and proves a weak rate of convergence of order 1 and an error expansion in the power of the time-step for the explicit family. Moreover, Alfonsi provides an interesting numerical comparison between the Deelstra and Delbaen scheme, his schemes and the present one discussed in this paper, in the special case of CIR processes.

In section 2, we construct our time discretisation scheme for $(X_t, t \in [0, T])$, based on the symmetrized Euler scheme and which can be simulated easily. We prove a theoretical rate of convergence of order one for the weak approximation error. We analyze separately the cases $\alpha = 1/2$ and $1/2 < \alpha < 1$. The convergence results are given in the next section in Theorems 2.3 and 2.5 respectively. The sections 3 and 4 are devoted to the proofs in this two respective situations. We denote $(\overline{X}_t, t \in [0, T])$ the approximation process. To study the weak error $\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T)$, we will use the Feynman–Kac representation $\mathbb{E}f(X_{T-t}^x) = u(t, x)$ where $u(t, x)$ solves the associated Kolmogorov PDE. The two main ingredients of the rate of convergence analysis consist in, first obtaining the upper-bounds on the spatial derivatives of $u(t, x)$ up to the order four. To our knowledge, for this kind of Cauchy problem, there is no generic result. The second point consists in studying the behavior of the approximation process at the origin.

Let us emphasis the difference between the situations $\alpha = 1/2$ and $1/2 < \alpha < 1$. The case $\alpha = 1/2$ could seem intuitively easier as the associated infinitesimal generator has unbounded but smooth coefficients. In fact, studying the spatial derivative of $u(t, x)$ with probabilistic tools, we need to impose the condition $b(0) > \sigma^2$, in order to define the

derivative of X_t^x with respect to x. In addition, the analysis of the approximation process $(\overline{X})$ at the origin shows that the expectation of its local time is in $\Delta t^{b(0)/\sigma^2}$.

In the case $1/2 < \alpha < 1$, the derivatives of the diffusion coefficient of the associated infinitesimal generator are degenerated functions at point zero. As we cannot hope to obtain uniform upper-bounds in x for the derivatives of $u(t,x)$, we prove that the approximation process goes to a neighborhood of the origin with an exponentially small probability and we give upper bounds for the negative moments of the approximation process $(\overline{X})$.

2 The symmetrized Euler scheme for (1)

For $x_0 \geq 0$, let $(X_t, t \geq 0)$ given by (1) or (2). For a fixed time $T > 0$, we define a discretisation scheme $(\overline{X}_{t_k}, k = 0, \ldots, N)$ by

$$\begin{cases} \overline{X}_0 = x_0 \geq 0, \\ \overline{X}_{t_{k+1}} = \left| \overline{X}_{t_k} + b(\overline{X}_{t_k})\Delta t + \sigma \overline{X}_{t_k}^{\alpha}(W_{t_{k+1}} - W_{t_k}) \right|, \end{cases} \tag{3}$$

$k = 0, \ldots, N-1$, where N denotes the number of discretisation times $t_k = k\Delta t$ and $\Delta t > 0$ is a constant time step such that $N\Delta t = T$.

In the sequel we will use the time continuous version $(\overline{X}_t, 0 \leq t \leq T)$ of the discrete time process, which consists in freezing the coefficients on each interval $[t_k, t_{k+1})$:

$$\overline{X}_t = \left| \overline{X}_{\eta(t)} + (t - \eta(t)) b(\overline{X}_{\eta(t)}) + \sigma \overline{X}_{\eta(t)}^{\alpha}(W_t - W_{\eta(t)}) \right|, \tag{4}$$

where $\eta(s) = \sup_{k \in \{1,\ldots,N\}} \{t_k; t_k \leq s\}$. The process $(\overline{X}_t, 0 \leq t \leq T)$ is valued in $[0, +\infty)$. By induction on each subinterval $[t_k, t_{k+1})$, for $k = 0$ to $N-1$, by using the Tanaka's formula, we can easily show that $(\overline{X}_t)$ is a continuous semi-martingale with a continuous local time $(L_t^0(\overline{X}))$ at point 0. Indeed, for any $t \in [0,T]$, if we set

$$\overline{Z}_t = \overline{X}_{\eta(t)} + b(\overline{X}_{\eta(t)})(t - \eta(t)) + \sigma \overline{X}_{\eta(t)}^{\alpha}(W_t - W_{\eta(t)}), \tag{5}$$

then $\overline{X}_t = |Z_t|$ and

$$\overline{X}_t = x_0 + \int_0^t \operatorname{sgn}(\overline{Z}_s) b(\overline{X}_{\eta(s)}) ds + \sigma \int_0^t \operatorname{sgn}(\overline{Z}_s) \overline{X}_{\eta(s)}^{\alpha} dW_s + \frac{1}{2} L_t^0(\overline{X}), \tag{6}$$

where $\operatorname{sgn}(x) := 1 - 2\,\mathbb{1}_{(x \leq 0)}$.

The following lemma ensures the existence of the positive moments of (X_t), starting at x_0 at time 0, and of $(\overline{X}_t)$, its associated discrete time process:

Lemma 2.1. *Assume* (H0) *and* (H1). *For any $x_0 \geq 0$, for any $p \geq 1$, there exists a positive constant C, depending on p, but also on the parameters $b(0)$, K, σ, α and T, such that*

$$\mathbb{E}\left(\sup_{t \in [0,T]} X_t^{2p} \right) + \mathbb{E}\left(\sup_{t \in [0,T]} \overline{X}_t^{2p} \right) \leq C(1 + x_0^{2p}). \tag{7}$$

In the following proof, as well as in the rest of the paper, C will denote a constant that can change from line to line. C could depend on the parameters of the model, but it is always independent of Δt.

Proof. We prove (7) for $(\overline{X}_t, 0 \leq t \leq T)$ only, the case of $(X_t, 0 \leq t \leq T)$ could be deduced by similar arguments. By the Itô's formula, and noting that for any $t \in [0,T]$ $\int_0^t \overline{X}_s^{2p-1} dL_s^0(\overline{X}) = 0$, we have

$$\begin{aligned} \overline{X}_t^{2p} &= x_0^{2p} + 2p \int_0^t \overline{X}_s^{2p-1} \operatorname{sgn}(\overline{Z}_s) b(\overline{X}_{\eta(s)}) ds \\ &+ 2p\sigma \textstyle\int_0^t \overline{X}_s^{2p-1} \operatorname{sgn}(\overline{Z}_s) \overline{X}_{\eta(s)}^{\alpha} dW_s + \sigma^2 p(2p-1) \displaystyle\int_0^t \overline{X}_s^{2p-2} \overline{X}_{\eta(s)}^{2\alpha} ds. \end{aligned} \tag{8}$$

To prove (7), let's start by showing that

$$\sup_{t\in[0,T]} \mathbb{E}\left(\overline{X}_t^{2p}\right) \le C(1+x_0^{2p}). \tag{9}$$

(7) will follow from (9), (8) and the Burkholder-Davis-Gundy Inequality. Let τ_n be the stopping time defined by $\tau_n = \inf\{0 < s < T; \overline{X}_s \ge n\}$, with $\inf\{\emptyset\} = 0$. Then,

$$\mathbb{E}\overline{X}_{t\wedge\tau_n}^{2p} \le x_0^{2p} + 2p\mathbb{E}\left(\int_0^{t\wedge\tau_n} \overline{X}_s^{2p-1} b(\overline{X}_{\eta(s)})ds\right) + \sigma^2 p(2p-1)\mathbb{E}\left(\int_0^{t\wedge\tau_n} \overline{X}_s^{2p-2}\overline{X}_{\eta(s)}^{2\alpha} ds\right).$$

By using (H0), (H1) and the Young Inequality, we get

$$\begin{aligned}\mathbb{E}\overline{X}_{t\wedge\tau_n}^{2p} \le & x_0^{2p} + Tb(0)^{2p} + (2p-1)\mathbb{E}\left(\int_0^{t\wedge\tau_n} \overline{X}_s^{2p} ds\right) \\ & + 2pK\mathbb{E}\left(\int_0^{t\wedge\tau_n} \overline{X}_s^{2p-1}\overline{X}_{\eta(s)} ds\right) + \sigma^2 p(2p-1)\mathbb{E}\left(\int_0^{t\wedge\tau_n} \overline{X}_s^{2p-2}\overline{X}_{\eta(s)}^{2\alpha} ds\right).\end{aligned}$$

Replacing $\overline{X}_s$ by (4) in the integrals above, by using another time (H1) and the Young Inequality, we easily obtain that for any $t \in [0,T]$,

$$\mathbb{E}\overline{X}_{\eta(t)\wedge\tau_n}^{2p} \le x_0^{2p} + C\left(1 + \int_0^{\eta(t)} \mathbb{E}\left(\overline{X}_{\eta(s)\wedge\tau_n}^{2p}\right) ds\right),$$

where $C > 0$ depends on p, $b(0)$, K, σ, α and T. A discrete version of the Gronwall Lemma allows us to conclude that

$$\sup_{k=0,\ldots,N} \mathbb{E}\left(\overline{X}_{t_k\wedge\tau_n}^{2p}\right) \le C(1+x_0^{2p}),$$

for another constant C, which does not depend on n. Taking the limit $n \to +\infty$, we get that $\sup_{k=0,\ldots,N} \mathbb{E}(\overline{X}_{t_k}^{2p}) \le C(1+x_0^{2p})$, from which we easily deduce (9) using (4). $\square$

2.1 Main results

In addition of hypotheses (H0) and (H1), we will analyze the convergence rate of (3) under the following hypothesis:

(H2) *The drift function $b(x)$ is a C^4 function, with bounded derivatives up to the order 4.*

2.1.1 Convergence rate when $\alpha = 1/2$

Under (H1), $(X_t, 0 \le t \le T)$ satisfies

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma\int_0^t \sqrt{X_s}dW_s, \quad 0 \le t \le T. \tag{10}$$

When $b(x)$ is of the form $a - \beta x$, with $a > 0$, (X_t) is the classical CIR process used in Financial mathematics to model the short interest rate. When $b(x) = a > 0$, (X_t) is the square of a Bessel process. Here we consider a generic drift function $b(x)$, with the following restriction :

(H3) $b(0) > \sigma^2$.

Remark 2.2. *When $x_0 > 0$ and $b(0) \ge \sigma^2/2$, by using the Feller's test, one can show that $\mathbb{P}(\tau_0 = \infty) = 1$ where $\tau_0 = \inf\{t \ge 0; X_t = 0\}$. We need the stronger Hypothesis* (H3) *to prove that the derivative (in the sense of the quadratic mean) of X_t^x with respect to x is well defined (see Proposition 3.4 and its proof in Appendix B). In particular, we need to use the Lemma 3.1 which controls the inverse moments and the exponential inverse moment of the CIR–like process (X_t), for some values of the parameter $\nu = \frac{2b(0)}{\sigma^2} - 1 > 1$.*

Section 3 is devoted to the proof of the following

Theorem 2.3. *Let f be a $\mathbb{R}$-valued C^4 bounded function, with bounded spatial derivatives up to the order 4. Let $\alpha = \frac{1}{2}$ and $x_0 > 0$. Assume* (H1), (H2) *and* (H3). *Choose Δt sufficiently small in* (3), *i.e.* $\Delta t \leq 1/(2K) \wedge x_0$. *Then there exists a positive constant C depending on f, b, T and x_0 such that*

$$\left|\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T)\right| \leq C \left(\Delta t + \left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right).$$

Under (H3), the global theoretical rate of convergence is of order one. When $b(0) < \sigma^2$, numerical tests for the CIR process show that the rate of convergence becomes under-linear (see [8] and the comparison of numerical schemes for the CIR process performed by Alfonsi in [1]).

2.1.2 Convergence rate when $1/2 < \alpha < 1$

Under (H1), $(X_t, 0 \leq t \leq T)$ satisfies

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma \int_0^t X_s^\alpha dW_s, \quad 0 \leq t \leq T. \tag{11}$$

We restrict ourselves to the case

(H3') $x_0 > \frac{b(0)}{\sqrt{2}} \Delta t$.

Remark 2.4. *When $x_0 > 0$, the Feller's test on process (X_t) shows that it is enough to suppose $b(0) > 0$, as in* (H1), *to ensure that $\mathbb{P}(\tau_0 = \infty) = 1$, for $\tau_0 = \inf\{t \geq 0; X_t = 0\}$.*

In Section 4, we prove the following

Theorem 2.5. *Let f be a $\mathbb{R}$-valued bounded C^4 function, with bounded spatial derivatives up to the order 4. Let $\frac{1}{2} < \alpha < 1$. Assume* (H1), (H2) *and* (H3'). *Choose Δt sufficiently small in* (3), *i.e.* $\Delta t \leq 1/(4K)$. *Then there exists a positive constant C depending on f, α, σ, b, T and x_0 such that*

$$|\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T)| \leq C \left(1 + \frac{1}{x_0^{q(\alpha)}}\right) \Delta t,$$

where $q(\alpha)$ is a positive constant depending only on α.

3 The case of processes with $\alpha = 1/2$

3.1 Preliminary results

In this section (X_t) denotes the solution of (10) starting at the deterministic point x_0 at time 0. When we need to vary the deterministic initial position, we mention it explicitly by using the notation (X_t^x) corresponding to the unique strong solution of the equation

$$X_t^x = x + \int_0^t b(X_s^x)ds + \sigma \int_0^t \sqrt{X_s^x} dW_s. \tag{12}$$

3.1.1 On the exact process

We have the following

Lemma 3.1. *Let us assume* (H1) *and* (H3). *We set* $\nu = \frac{2b(0)}{\sigma^2} - 1 > 1$. *For any* p *such that* $1 < p < \nu$, *for any* $t \in [0, T]$ *and any* $x > 0$,

$$\mathbb{E}\left(X_t^x\right)^{-1} \leq C(T)x^{-1} \text{ and } \mathbb{E}\left(X_t^x\right)^{-p} \leq C(T)t^{-p} \text{ or } \mathbb{E}\left(X_t^x\right)^{-p} \leq C(T,p)x^{-p}.$$

Moreover for all $\mu \leq \frac{\nu^2\sigma^2}{8}$,

$$\mathbb{E}\exp\left(\mu\int_0^t (X_s^x)^{-1}ds\right) \leq C(T)\left(1 + x^{-\nu/2}\right), \tag{13}$$

where the positive constant $C(T)$ *is a non-decreasing function of* T *and does not depend on* x.

Proof. As $b(x) \geq b(0) - Kx$, The Comparison Theorem gives that, a.s. $X_t^x \geq Y_t^x$, for all $t \geq 0$, where $(Y_t^x, t \leq T)$ is the CIR process solving

$$Y_t^x = x + \int_0^t \left(b(0) - KY_s^x\right) ds + \sigma\int_0^t \sqrt{Y_s^x}dW_s. \tag{14}$$

In particular, $\mathbb{E}\exp(\mu\int_0^t (X_s^x)^{-1}ds) \leq \mathbb{E}\exp(\mu\int_0^t (Y_s^x)^{-1}ds)$. As $b(0) > \sigma^2$ by (H3), one can derive (13) from the Lemma A.2. Similarly, for the upper bounds on the inverse moments of X_t^x, we apply the Lemma A.1. □

3.1.2 On the associated Kolmogorov PDE

Proposition 3.2. *Let* $\alpha = 1/2$. *Let* f *be a* $\mathbb{R}$*-valued* C^4 *bounded function, with bounded spatial derivatives up to the order* 4. *We consider the* $\mathbb{R}$*-valued function defined on* $[0, T] \times [0, +\infty)$ *by* $u(t, x) = \mathbb{E}f(X_{T-t}^x)$. *Under* (H1), (H2) *and* (H3), u *is in* $C^{1,4}([0, T] \times [0, +\infty))$. *That is,* u *has a first derivative in the time variable and derivatives up to order 4 in the space variable. Moreover, there exists a positive constant* C *depending on* f, b *and* T *such that, for all* $x \in [0, +\infty)$,

$$\sup_{t\in[0,T]}\left|\frac{\partial u}{\partial t}(t, x)\right| \leq C(1 + x), \tag{15}$$

$$\|u\|_{L^\infty([0,T]\times[0,+\infty))} + \sum_{k=1}^{4}\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L^\infty([0,T]\times[0,+\infty))} \leq C \tag{16}$$

and $u(t, x)$ *satisfies*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + b(x)\dfrac{\partial u}{\partial x}(t, x) + \dfrac{\sigma^2}{2}x\dfrac{\partial^2 u}{\partial x^2}(t, x) = 0, \ (t, x) \in [0, T) \times [0, +\infty), \\ u(T, x) = f(x), \ x \in [0, +\infty). \end{cases} \tag{17}$$

In all what follows, we will denote $\| \ \|_\infty$ the norm on L^∞ spaces. Before to prove the Proposition 3.2, we introduce some notations and give preliminary results: for any $\lambda \geq 0$ and any $x > 0$, we denote by $(X_t^x(\lambda), 0 \leq t \leq T)$, the $[0, +\infty)$-valued process solving

$$X_t^x(\lambda) = x + \lambda\sigma^2 t + \int_0^t b(X_s^x(\lambda))ds + \sigma\int_0^t \sqrt{X_s^x(\lambda)}dW_s. \tag{18}$$

Equation (18) has a non-exploding unique strong solution. Moreover, for all $t \geq 0$, $X_t^x(\lambda) \geq X_t^x$. The coefficients are locally Lipschitz on $(0, +\infty)$, with locally Lipschitz first order derivatives. Then $X_t^x(\lambda)$ is continuously differentiable

(see e.g. Theorem V.39 in [16]), and if we denote $J_t^x(\lambda) = \frac{dX_t^x}{dx}(\lambda)$, the process $(J_t^x(\lambda), 0 \le t \le T)$ satisfies the linear equation

$$J_t^x(\lambda) = 1 + \int_0^t J_s^x(\lambda) b'(X_s^x(\lambda))ds + \int_0^t J_s^x(\lambda) \frac{\sigma dW_s}{2\sqrt{X_s^x(\lambda)}}. \tag{19}$$

By Lemma 3.1, the process $(\int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}}, 0 \le t \le T)$ is a locally square integrable martingale. Then, for all $\lambda \ge 0$, $J_t^x(\lambda)$ is given by (see e.g. Theorem V.51 in [16]),

$$J_t^x(\lambda) = \exp\left(\int_0^t b'(X_s^x(\lambda))ds + \frac{\sigma}{2}\int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X_s^x(\lambda)} \right). \tag{20}$$

Lemma 3.3. *Assume* (H3). *The process* $(M_t^x(\lambda), 0 \le t \le T)$ *defined by*

$$M_t^x(\lambda) = \exp\left(\frac{\sigma}{2}\int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X_s^x(\lambda)} \right)$$

is a $\mathbb{P}$*-martingale. Moreover,* $\sup_{t\in[0,T]} \mathbb{E}\left(J_t^x(\lambda)\right) \le C$ *where the positive constant* C *does not depend on* x *.*

Proof. By Lemma 3.1, $(M_t^x(\lambda), 0 \le t \le T)$ satisfies the Novikov criterion. Under (H2), b' is a bounded function and

$$\mathbb{E}\left[J_t^x(\lambda)\right] = \mathbb{E}\left[\exp\left(\int_0^t b'(X_s^x(\lambda))ds\right) M_t^x(\lambda)\right] \le \exp(\|b'\|_\infty T).$$

□

Let $(\mathcal{Z}_t^{(\lambda,\lambda+\frac{1}{2})}, 0 \le t \le T)$ defined by

$$\begin{aligned}\mathcal{Z}_t^{(\lambda,\lambda+\frac{1}{2})} = \quad & \exp\left(-\tfrac{\sigma}{2}\int_0^t \tfrac{1}{\sqrt{X_s^x(\lambda)}} \left(\tfrac{dX_s^x(\lambda)}{\sigma\sqrt{X_s^x(\lambda)}} \right.\right. \\ & \left.\left. -\tfrac{b(X_s^x(\lambda))+(\lambda+\frac{1}{2})\sigma^2}{\sigma\sqrt{X_s^x(\lambda)}} ds \right) \tfrac{\sigma^2}{8}\int_0^t \tfrac{ds}{X_s^x(\lambda)} \right).\end{aligned} \tag{21}$$

By the Girsanov Theorem, under the probability $\mathbb{Q}^{\lambda+\frac{1}{2}}$ such that $\frac{d\mathbb{Q}^{\lambda+\frac{1}{2}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{1}{Z_t^{(\lambda,\lambda+\frac{1}{2})}}$, the process $(B_t^{\lambda+\frac{1}{2}} = \int_0^t \frac{dX_s^x(\lambda)}{\sigma\sqrt{X_s^x(\lambda)}} - \frac{b(X_s^x(\lambda))+(\lambda+\frac{1}{2})\sigma^2}{\sigma\sqrt{X_s^x(\lambda)}}ds, t \in [0,T])$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{Q}^{\lambda+\frac{1}{2}})$. Indeed, we have that

$$X_t^x(\lambda) = x + (\lambda + \frac{1}{2})\sigma^2 t + \int_0^t b(X_s^x(\lambda))ds + \sigma\int_0^t \sqrt{X_s^x(\lambda)}dB_s^{\lambda+\frac{1}{2}}.$$

Hence, $\mathcal{L}^{\mathbb{Q}^{\lambda+\frac{1}{2}}}(X_\cdot^x(\lambda)) = \mathcal{L}^{\mathbb{P}}(X_\cdot^x(\lambda+\frac{1}{2}))$ and from the Lemma 3.3, $\mathcal{Z}_t^{(\lambda,\lambda+\frac{1}{2})} = \exp(-\frac{\sigma}{2}\int_0^t \frac{dB_s^{\lambda+\frac{1}{2}}}{\sqrt{X_s(\lambda)}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X_s(\lambda)})$ is a $\mathbb{Q}^{\lambda+\frac{1}{2}}$–martingale. The following proposition allows us to compute the derivatives of $u(t,x)$.

Proposition 3.4. *Assume* (H1), (H2) *and* (H3). *Let* $g(x)$, $h(x)$ *and* $k(x)$ *be some bounded* C^1 *functions, with bounded first derivatives. For any* $\lambda \ge 0$, *let* $v(t,x)$ *be the* $\mathbb{R}$*-valued function defined, on* $[0,T] \times \mathbb{R}_+^*$, *by*

$$v(t,x) = \mathbb{E}\left[g(X_t^x(\lambda))\exp(\int_0^t k(X_s^x(\lambda))ds)\right] + \int_0^t \mathbb{E}\left[h(X_s^x(\lambda))\exp(\int_0^s k(X_\theta^x(\lambda))d\theta)\right]ds.$$

Then $v(t,x)$ *is of class* C^1 *with respect to* x *and*

$$\begin{aligned}\frac{\partial v}{\partial x}(t,x) &= \mathbb{E}\left[\exp(\int_0^t k(X_s^x(\lambda))ds)\left(g'(X_t^x(\lambda))J_t^x(\lambda) + g(X_t^x(\lambda))\int_0^t k'(X_s^x(\lambda))J_s^x(\lambda)ds\right)\right] \\ &+ \int_0^t \mathbb{E}\left[\exp(\int_0^s k(X_\theta^x(\lambda))d\theta)\left(h'(X_s^x(\lambda))J_s^x(\lambda) + h(X_s^x(\lambda))\int_0^s k'(X_\theta^x(\lambda))J_\theta^x(\lambda)d\theta\right)\right]ds.\end{aligned}$$

The proof is postponed in the Appendix B.

Proof of Proposition 3.2. First, we note that $u(t,x) = \mathbb{E}f(X^x_{T-t})$ is a continuous function in x and bounded by $\|f\|_\infty$. Let us show that u is in $C^{1,4}([0,T]\times[0,+\infty))$. f being in $C^4(\mathbb{R})$, by the Itô's formula,

$$\begin{aligned} u(t,x) = f(x) &+ \int_0^{T-t} \mathbb{E}\left(b(X^x_s)f'(X^x_s)\right)ds + \frac{\sigma^2}{2}\int_0^{T-t}\mathbb{E}\left(X^x_s f''(X^x_s)\right)ds \\ &+ \sigma\mathbb{E}\left(\int_0^{T-t}\sqrt{X^x_s}f'(X^x_s)dW_s\right). \end{aligned}$$

f' is bounded and (X^x_t) have moments of any order. Then we obtain that

$$\frac{\partial u}{\partial t}(t,x) = -\mathbb{E}\left(b(X^x_{T-t})f'(X^x_{T-t}) + \frac{\sigma^2}{2}X^x_{T-t}f''(X^x_{T-t})\right).$$

Hence, $\frac{\partial u}{\partial t}$ is a continuous function on $[0,T]\times[0,+\infty)$ and (15) follows by Lemma 2.1.

By Proposition 3.4, for $x>0$, $u(t,x) = \mathbb{E}f(X^x_{T-t})$ is differentiable and

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}\left(f'(X^x_{T-t}(0))J^x_{T-t}(0)\right).$$

Hence, by using the Lemma 3.3, $\left|\frac{\partial u}{\partial x}(t,x)\right| \le \|f'\|_\infty \mathbb{E}\left(J^x_{T-t}(0)\right) \le C\|f'\|_\infty$. We introduce the probability $\mathbb{Q}^{\frac{1}{2}}$ such that $\left.\frac{d\mathbb{Q}^{\frac{1}{2}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t^{(0,\frac{1}{2})}}$. Denoting by $\mathbb{E}^{\frac{1}{2}}$ the expectation under the probability $\mathbb{Q}^{\frac{1}{2}}$, we have

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}^{\frac{1}{2}}\left(f'(X^x_{T-t}(0))\mathcal{Z}^{(0,\frac{1}{2})}_{T-t}J^x_{T-t}(0)\right).$$

From (20), as $W_t = B^{\frac{1}{2}}_t + \int_0^t \frac{\sigma}{2\sqrt{X^x_s(0)}}ds$, we notice that

$$J^x_t(0) = \exp\left(\int_0^t b'(X^x_s(0)ds + \frac{\sigma}{2}\int_0^t \frac{dB^{\frac{1}{2}}_s}{\sqrt{X^x_s(0)}} + \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_s(0)}\right)$$

and that $\mathcal{Z}^{(0,\frac{1}{2})}_{T-t}J^x_{T-t}(0) = \exp\left(\int_0^{T-t}b'(X^x_s(0))ds\right)$, from the definition of $\mathcal{Z}^{(0,\frac{1}{2})}_t$ in (21). Hence, $\frac{\partial u}{\partial x}(t,x) = \mathbb{E}^{\frac{1}{2}}\left[f'(X^x_{T-t}(0))\exp\left(\int_0^{T-t}b'(X^x_s(0))ds\right)\right]$. As $\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x_\cdot(0)) = \mathcal{L}^{\mathbb{P}}(X^x_\cdot(\frac{1}{2}))$, for $x>0$, we finally obtain the following expression for $\frac{\partial u}{\partial x}(t,x)$:

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}\left[f'(X^x_{T-t}(\frac{1}{2}))\exp\left(\int_0^{T-t}b'(X^x_s(\frac{1}{2}))ds\right)\right]. \tag{22}$$

Now the right-hand side of (22) is a continuous function on $[0,T]\times[0,+\infty)$ so that $u\in C^{1,1}([0,T]\times[0,+\infty))$. Moreover for $x>0$, by Proposition 3.4, $\frac{\partial u}{\partial x}(t,x)$ is continuously differentiable and

$$\begin{aligned} \tfrac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left[f''(X^x_{T-t}(\frac{1}{2}))J^x_{T-t}(\frac{1}{2})\exp\left(\int_0^{T-t}b'(X^x_s(\frac{1}{2}))ds\right)\right] \\ +\mathbb{E}&\left[f'(X^x_{T-t}(\frac{1}{2}))\exp\left(\int_0^{T-t}b'(X^x_s(\frac{1}{2}))ds\right)\left(\int_0^{T-t}b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})ds\right)\right]. \end{aligned} \tag{23}$$

As previously, we can conclude that $\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right|$ is bounded uniformly in x. In order to obtain an expression for $\frac{\partial^2 u}{\partial x^2}(t,x)$ continuous in $[0,T]\times[0,+\infty)$ and also to compute the third derivative, we need to transform the expression in (23) in order to avoid again the appearance of the derivative of $J^x_t(\frac{1}{2})$ that we do not control. Thanks to the Markov property and the time homogeneity of the process $X^x_t(\frac{1}{2}))$, for any $s\in[0,T-t]$,

$$\begin{aligned}
&\mathbb{E}\left[f'(X^x_{T-t}(\frac{1}{2}))\exp\left(\int_s^{T-t} b'(X^x_u(\frac{1}{2}))\right)\Big/\mathcal{F}_s\right]\\
&=\mathbb{E}\left[f'(X^y_{T-t-s}(\frac{1}{2}))\exp\left(\int_0^{T-t-s} b'(X^y_u(\frac{1}{2}))\right)\right]\Big|_{y=X^x_s(\frac{1}{2}))}.
\end{aligned}$$

By using (22), we get $\mathbb{E}[f'(X^x_{T-t}(\frac{1}{2}))\exp(\int_s^{T-t} b'(X^x_u(\frac{1}{2})))/\mathcal{F}_s] = \frac{\partial u}{\partial x}(t+s,X^x_s(\frac{1}{2})))$. We introduce this last equality in the second term of the right-hand side of (23):

$$\begin{aligned}
&\mathbb{E}\left[f'(X^x_{T-t}(\frac{1}{2}))\exp\left(\int_0^{T-t} b'(X^x_u(\frac{1}{2}))du\right)\left(\int_0^{T-t} b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})ds\right)\right]\\
&=\mathbb{E}\left[\int_0^{T-t}\mathbb{E}\left(f'(X^x_{T-t}(\frac{1}{2}))\exp\left(\int_s^{T-t} b'(X^x_u(\frac{1}{2}))du\right)\Big/\mathcal{F}_s\right)\right.\\
&\qquad\qquad\left.\times\exp\left(\int_0^s b'(X^x_u(\frac{1}{2}))du\right)b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})ds\right]\\
&=\int_0^{T-t}\mathbb{E}\left[\frac{\partial u}{\partial x}(t+s,X^x_s(\frac{1}{2})))\exp\left(\int_0^s b'(X^x_u(\frac{1}{2}))du\right)b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})\right]ds.
\end{aligned}$$

Coming back to (23), this leads to the following expression for $\frac{\partial^2 u}{\partial x^2}(t,x)$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x)&=\mathbb{E}\left[f''(X^x_{T-t}(\frac{1}{2}))J^x_{T-t}(\frac{1}{2})\exp\left(\int_0^{T-t} b'(X^x_s(\frac{1}{2}))ds\right)\right]\\
&+\int_0^{T-t}\mathbb{E}\left[\frac{\partial u}{\partial x}(t+s,X^x_s(\frac{1}{2})))\exp\left(\int_0^s b'(X^x_u(\frac{1}{2}))du\right)b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})\right]ds.
\end{aligned}$$

We introduce the probability $\mathbb{Q}^1$ such that $\frac{d\mathbb{Q}^1}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t^{(\frac{1}{2},1)}}$. Then,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x)&=\mathbb{E}^1\left[\mathcal{Z}^{(\frac{1}{2},1)}_{T-t}f''(X^x_{T-t}(\frac{1}{2}))J^x_{T-t}(\frac{1}{2})\exp\left(\int_0^{T-t} b'(X^x_s(\frac{1}{2}))ds\right)\right]\\
&+\int_0^{T-t}\mathbb{E}^1\left[\mathcal{Z}^{(\frac{1}{2},1)}_s\frac{\partial u}{\partial x}(t+s,X^x_s(\frac{1}{2})))\exp\left(\int_0^s b'(X^x_u(\frac{1}{2}))du\right)b''(X^x_s(\frac{1}{2}))J^x_s(\frac{1}{2})\right]ds.
\end{aligned}$$

Again for all $\theta\in[0,T]$, we have that $\mathcal{Z}^{(\frac{1}{2},1)}_\theta J^x_\theta(\frac{1}{2}) = \exp\left(\int_0^\theta b'(X^x_u(\frac{1}{2}))du\right)$ and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x)=&\mathbb{E}^1\left[f''(X^x_{T-t}(\frac{1}{2}))\exp\left(2\int_0^{T-t} b'(X^x_s(\frac{1}{2}))ds\right)\right]\\
&+\int_0^{T-t}\mathbb{E}^1\left[\frac{\partial u}{\partial x}(t+s,X^x_s(\frac{1}{2})))\exp\left(2\int_0^s b'(X^x_u(\frac{1}{2}))du\right)b''(X^x_s(\frac{1}{2}))\right]ds.
\end{aligned}$$

As $\mathcal{L}^{\mathbb{Q}^1}(X_\cdot^x(\frac{1}{2})) = \mathcal{L}^{\mathbb{P}}(X_\cdot^x(1))$, we finally obtain the following expression for $\frac{\partial^2 u}{\partial x^2}(t,x)$:

$$
\begin{aligned}
\tfrac{\partial^2 u}{\partial x^2}(t,x) = \quad & \mathbb{E}\left[f''(X_{T-t}^x(1))\exp\left(2\int_0^{T-t} b'(X_s^x(1))ds\right)\right] \\
& + \textstyle\int_0^{T-t}\mathbb{E}\left[\dfrac{\partial u}{\partial x}(t+s, X_s^x(1)))\exp\left(2\displaystyle\int_0^s b'(X_u^x(1))du\right)b''(X_s^x(1))\right]ds
\end{aligned}
\tag{24}
$$

from which, we deduce that $u \in C^{1,2}([0,T]\times[0,+\infty))$. As $J_s^x(1) = \frac{dX_s^x(1)}{dx}$ exists and is given by (20), for $x > 0$, $\frac{\partial^2 u}{\partial x^2}(t,x)$ is continuously differentiable (see again Proposition 3.4) and

$$
\begin{aligned}
\tfrac{\partial^3 u}{\partial x^3}(t,x) = \quad & \mathbb{E}\left\{\exp\left(2\int_0^{T-t} b'(X_s^x(1))ds\right)\right. \\
& \quad \left.\times\left[f^{(3)}(X_{T-t}^x(1))J_{T-t}^x(1) + 2f''(X_{T-t}^x(1))\textstyle\int_0^{T-t} b''(X_s^x(1))J_s^x(1)ds\right]\right\} \\
& + \textstyle\int_0^{T-t}\mathbb{E}\left\{\exp\left(2\displaystyle\int_0^s b'(X_u^x(1)du\right)\right. \\
& \quad \times\left[J_s^x(1)\left(\tfrac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)))b''(X_s^x(1)) + \tfrac{\partial u}{\partial x}(t+s, X_s^x(1)))b^{(3)}(X_s^x(1))\right)\right. \\
& \quad\quad \left.\left. + 2\tfrac{\partial u}{\partial x}(t+s, X_s^x(1)))b''(X_s^x(1))\textstyle\int_0^s b''(X_u^x(1))J_u^x(1)du\right]\right\}ds.
\end{aligned}
\tag{25}
$$

By Lemma 3.3, the derivatives of f and b being bounded up to the order 3, we get immediately that $|\frac{\partial^3 u}{\partial x^3}(t,x)| \leq C$ uniformly in x.

The computation of the fourth derivative uses similar arguments. We detail it in the Appendix C.

In view of (15) and (16), one can adapt easily the proof of the Theorem 6.1 in [10] and show that $u(t,x)$ solves the Cauchy problem (17). ☐

3.1.3 On the approximation process

According to (3) and (6), the discrete time process $(\overline{X})$ associated to (X) is

$$
\begin{cases}
\overline{X}_0 = x_0, \\
\overline{X}_{t_{k+1}} = \left|\overline{X}_{t_k} + b(\overline{X}_{t_k})\Delta t + \sigma\sqrt{\overline{X}_{t_k}}(W_{t_{k+1}} - W_{t_k})\right|,
\end{cases}
\tag{26}
$$

$k = 0, \ldots, N-1$, and the time continuous version $(\overline{X}_t, 0 \leq t \leq T)$ satisfies

$$
\overline{X}_t = x_0 + \int_0^t \operatorname{sgn}(\overline{Z}_s)b(\overline{X}_{\eta(s)})ds + \sigma\int_0^t \operatorname{sgn}(\overline{Z}_s)\sqrt{\overline{X}_{\eta(s)}}dW_s + \frac{1}{2}L_t^0(\overline{X}),
\tag{27}
$$

where $\operatorname{sgn}(x) = 1 - 2\,\mathbb{1}_{(x\leq 0)}$, and for any $t \in [0,T]$,

$$
\overline{Z}_t = \overline{X}_{\eta(t)} + (t-\eta(t))b(\overline{X}_{\eta(t)}) + \sigma\sqrt{\overline{X}_{\eta(t)}}(W_t - W_{\eta(t)}).
\tag{28}
$$

In this section, we are interested in the behavior of the processes $(\overline{X})$ and $(\overline{Z})$ visiting the point 0. The main result is the following

Proposition 3.5. *Let $\alpha = \frac{1}{2}$. Assume* (H1). *For Δt sufficiently small ($\Delta t \leq 1/(2K)$), there exists a constant $C > 0$, depending on $b(0)$, K, σ, x_0 and T but not in Δt, such that*

$$
\begin{aligned}
& \mathbb{E}\left(L_t^0(\overline{X}) - L_{\eta(t)}^0(\overline{X})\Big/\mathcal{F}_{\eta(t)}\right) \leq C\Delta t\exp\left(-\frac{\overline{X}_{\eta(t)}}{16\sigma^2\Delta t}\right) \\
& \qquad\qquad \textit{and}\ \ \mathbb{E}L_T^0(\overline{X}) \leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.
\end{aligned}
\tag{29}
$$

The upper bounds above, for the local time $(L^0_\cdot(\overline{X}))$, are based on the following technical lemmas:

Lemma 3.6. *Assume* (H1). *Assume also that Δt is sufficiently small ($\Delta t \leq 1/(2K) \wedge x_0$). Then for any $\gamma \geq 1$, there exists a positive constant C, depending on all the parameters $b(0)$, K, σ, x_0, T and also on γ, such that*

$$\sup_{k=0,\dots,N} \mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})}.$$

Lemma 3.7. *Assume* (H1). *For Δt sufficiently small ($\Delta t \leq 1/(2K)$), for any $t \in [0,T]$,*

$$\mathbb{P}\left(\overline{Z}_t \leq 0/\,\overline{X}_{\eta(t)}\right) \leq \frac{1}{2}\exp\left(-\frac{\overline{X}_{\eta(t)}}{2(1-K\Delta t)^{-2}\,\sigma^2\Delta t}\right).$$

As $2(1-K\Delta t)^{-2} > 1$ when $\Delta t \leq 1/(2K)$, the combination of Lemmas 3.6 and 3.7 leads to

$$\mathbb{P}\left(\overline{Z}_t \leq 0\right) \leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}. \tag{30}$$

We give successively the proofs of Lemmas 3.6, 3.7 and Proposition 3.5.

Proof of Lemma 3.6. First, we show that there exits a positive sequence $(\mu_j, 0 \leq j \leq N)$ such that, for any $k \in \{1,\dots,N\}$,

$$\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq \exp\left(-b(0)\sum_{j=0}^{k-1}\mu_j\Delta t\right)\exp\left(-x_0\mu_k\right).$$

We set $\mu_0 = \frac{1}{\gamma\sigma^2\Delta t}$. By (26), as $-b(x) \leq -b(0) + Kx$, for all $x \in \mathbb{R}$, we have that

$$\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_k}\right) \leq \mathbb{E}\exp\left(-\mu_0\left(\overline{X}_{t_{k-1}} + (b(0) - K\overline{X}_{t_{k-1}})\Delta t + \sigma\sqrt{\overline{X}_{t_{k-1}}}\Delta W_{t_k}\right)\right).$$

ΔW_{t_k} and $\overline{X}_{t_{k-1}}$ being independent, $\mathbb{E}\exp(-\mu_0\sigma\sqrt{\overline{X}_{t_{k-1}}}\Delta W_{t_k}) = \mathbb{E}\exp(\frac{\sigma^2}{2}\mu_0^2\Delta t\overline{X}_{t_{k-1}})$. Thus

$$\begin{aligned}\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_k}\right) &\leq \exp\left(-\mu_0 b(0)\Delta t\right)\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_{k-1}}\left(1 - K\Delta t - \frac{\sigma^2}{2}\mu_0\Delta t\right)\right)\\ &= \exp\left(-\mu_0 b(0)\Delta t\right)\mathbb{E}\exp\left(-\mu_1\overline{X}_{t_{k-1}}\right),\end{aligned}$$

where we set $\mu_1 = \mu_0(1 - K\Delta t - \frac{\sigma^2}{2}\mu_0\Delta t)$. Consider now the sequence $(\mu_j)_{i\in\mathbb{N}}$ defined by

$$\begin{cases}\mu_0 = \frac{1}{\gamma\sigma^2\Delta t},\\ \mu_j = \mu_{j-1}\left(1 - K\Delta t - \frac{\sigma^2}{2}\mu_{j-1}\Delta t\right),\ j \geq 1.\end{cases} \tag{31}$$

An easy computation shows that if $\gamma \geq 1$ and $\Delta t \leq \frac{1}{2K}$, then (μ_j) is a positive and decreasing sequence. For any $j \in \{0,\dots,k-1\}$, by the same computation we have

$$\mathbb{E}\exp\left(-\mu_j\overline{X}_{t_{k-j}}\right) \leq \exp\left(-b(0)\mu_j\Delta t\right)\mathbb{E}\exp\left(-\mu_{j+1}\overline{X}_{t_{k-j-1}}\right)$$

and it follows by induction that

$$\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_k}\right) \leq \exp\left(-b(0)\sum_{j=0}^{k-1}\mu_j\Delta t\right)\exp\left(-x_0\mu_k\right). \tag{32}$$

Now, we study the sequence $(\mu_j, 0 \le j \le N)$. For any $\alpha > 0$, we consider the non-decreasing function $f_\alpha(x) := \frac{x}{1+\alpha x}$, $x \in \mathbb{R}$. We note that $(f_\alpha \circ f_\beta)(x) = f_{\alpha+\beta}(x)$. Then, for any $j \ge 1$, the sequence (μ_j) being decreasing, $\mu_j \le \mu_{j-1} - \frac{\sigma^2}{2}\Delta t \mu_j \mu_{j-1}$, and

$$\mu_j \le f_{\frac{\sigma^2}{2}\Delta t}(\mu_{j-1}) \le f_{\frac{\sigma^2}{2}\Delta t}\left(f_{\frac{\sigma^2}{2}\Delta t}(\mu_{j-2})\right) \le \ldots \le f_{\frac{\sigma^2}{2}j\Delta t}(\mu_0). \tag{33}$$

The next step consists in proving, by induction, the following lower bound for the μ_j:

$$\mu_j \ge \mu_1 \left(\frac{1}{1+\frac{\sigma^2}{2}\Delta t(j-1)\mu_0}\right) - K\left(\frac{\Delta t(j-1)\mu_0}{1+\frac{\sigma^2}{2}\Delta t(j-1)\mu_0}\right),\ \forall j \ge 1. \tag{34}$$

(34) is clearly true for $j = 1$. Suppose (34) holds for j. By (31) and (33)

$$\begin{aligned}\mu_{j+1} = \mu_j\left(1 - \frac{\sigma^2}{2}\Delta t \mu_j\right) - K\Delta t \mu_j \ge& \mu_j\left(1 - \frac{\sigma^2}{2}\Delta t f_{\frac{\sigma^2}{2}j\Delta t}(\mu_0)\right) - K\Delta t f_{\frac{\sigma^2}{2}j\Delta t}(\mu_0)\\ \ge& \mu_j\left(\frac{1+\frac{\sigma^2}{2}\Delta t(j-1)\mu_0}{1+\frac{\sigma^2}{2}\Delta t j\mu_0}\right) - K\left(\frac{\Delta t\mu_0}{1+\frac{\sigma^2}{2}\Delta t j\mu_0}\right)\end{aligned}$$

and we conclude by using (34) for μ_j. Now, we replace μ_0 by its value $\frac{1}{\gamma\sigma^2\Delta t}$ in (34) and obtain that $\mu_j \ge \frac{2\gamma-1}{\Delta t\gamma\sigma^2(2\gamma-1+j)} - \frac{2K}{\sigma^2}$, for any $j \ge 0$. Hence,

$$\begin{aligned}\sum_{j=0}^{k-1}\Delta t\mu_j \ge \frac{1}{\gamma\sigma^2}\sum_{j=0}^{k-1}\frac{2\gamma-1}{2\gamma-1+j} - \frac{2Kt_k}{\sigma^2} \ge \frac{1}{\gamma\sigma^2}\int_0^k \frac{2\gamma-1}{2\gamma-1+u}du - \frac{2KT}{\sigma^2}\\ \ge \frac{2\gamma-1}{\gamma\sigma^2}\ln\left(\frac{2\gamma-1+k}{2\gamma-1}\right) - \frac{2KT}{\sigma^2}.\end{aligned}$$

Coming back to (32), we obtain that

$$\begin{aligned}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \le& \exp\left(b(0)\frac{2KT}{\sigma^2}\right)\exp\left(x_0\frac{2K}{\sigma^2}\right)\\ &\times\left(\frac{2\gamma-1}{2\gamma-1+k}\right)^{\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})}\exp\left(-\frac{x_0}{\Delta t\gamma\sigma^2}\frac{(2\gamma-1)}{(2\gamma-1+k)}\right).\end{aligned}$$

Finally, we use the inequality $x^\alpha\exp(-x) \le \alpha^\alpha\exp(-\alpha)$, for all $\alpha > 0$ and $x > 0$. It comes that

$$\begin{aligned}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \le& \exp\left(b(0)\frac{2KT}{\sigma^2}\right)\exp\left(x_0\frac{2K}{\sigma^2}\right)\\ &\times\left(2b(0)\frac{\Delta t\gamma}{x_0}(1-\frac{1}{2\gamma})\right)^{\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})}\exp\left(-\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})\right)\\ \le& C\left(\frac{\Delta t}{x_0}\right)^{\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})}\end{aligned}$$

where we set

$$C = (b(0)(2\gamma-1))^{\frac{2b(0)}{\sigma^2}(1-\frac{1}{2\gamma})}\exp\left(\frac{2}{\sigma^2}\left(b(0)(KT-1+\frac{1}{2\gamma})+x_0K\right)\right).$$

□

Proof of Lemma 3.7. Under (H1), $b(x) \geq b(0) - Kx$, for $x \geq 0$. Then, by the definition of $(\overline{Z})$ in (28),

$$\mathbb{P}\left(\overline{Z}_t \leq 0\right) \leq \mathbb{P}\left(W_t - W_{\eta(t)} \leq \frac{-\overline{X}_{\eta(t)}(1 - K(t-\eta(t))) - b(0)(t-\eta(t))}{\sigma\sqrt{\overline{X}_{\eta(t)}}}, \overline{X}_{\eta(t)} > 0\right).$$

By using the Gaussian inequality $\mathbb{P}(G \leq \beta) \leq 1/2\exp(-\beta^2/2)$, for a standard Normal r.v. G and $\beta < 0$, we get

$$\mathbb{P}\left(\overline{Z}_t \leq 0\right) \leq \frac{1}{2}\mathbb{E}\left[\exp\left(-\frac{(\overline{X}_{\eta(t)}(1-K(t-\eta(t))) + b(0)(t-\eta(t)))^2}{2\sigma^2(t-\eta(t))\overline{X}_{\eta(t)}}\right)\mathbb{1}_{\{\overline{X}_{\eta(t)}>0\}}\right]$$

from which we finally obtain that $\mathbb{P}\left(\overline{Z}_t \leq 0\right) \leq \frac{1}{2}\mathbb{E}[\exp(-\frac{\overline{X}_{\eta(t)}}{2(1-K\Delta t)^{-2}\sigma^2\Delta t})]$. ☐

Proof of Proposition 3.5. By the occupation time formula, for any $t \in [t_k, t_{k+1})$, $0 \leq k \leq N$, for any bounded Borel-measurable function ϕ, $\mathbb{P}$ a.s

$$\begin{aligned}\int_{\mathbb{R}} \phi(z)\left(L_t^z(\overline{X}) - L_{t_k}^z(\overline{X})\right)dz &= \int_{\mathbb{R}} \phi(z)\left(L_t^z(\overline{Z}) - L_{t_k}^z(\overline{Z})\right)dz \\ &= \int_{t_k}^t \phi(\overline{Z}_s)d\langle\overline{Z},\overline{Z}\rangle_s = \sigma^2\int_{t_k}^t \phi(\overline{Z}_s)\overline{X}_{t_k}ds.\end{aligned}$$

Hence, for any $x > 0$, an easy computation shows that

$$\begin{aligned}&\int_{\mathbb{R}} \phi(z)\mathbb{E}\left(L_t^z(\overline{X}) - L_{t_k}^z(\overline{X})/\left\{\overline{X}_{t_k} = x\right\}\right)dz = \sigma^2\int_{t_k}^t x\mathbb{E}\left(\phi(\overline{Z}_s)/\left\{\overline{X}_{t_k} = x\right\}\right)ds \\ &= \sigma\int_{\mathbb{R}}\phi(z)\int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}}\exp\left(-\frac{(z-x-b(x)(s-t_k))^2}{2\sigma^2 x(s-t_k)}\right)ds\,dz.\end{aligned}$$

Then, for any $z \in \mathbb{R}$,

$$\mathbb{E}\left(L_t^z(\overline{X}) - L_{t_k}^z(\overline{X})/\left\{\overline{X}_{t_k} = x\right\}\right) = \sigma\int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}}\exp\left(-\frac{(z-x-b(x)(s-t_k))^2}{2\sigma^2 x\,(s-t_k)}\right)ds.$$

In particular for $z = 0$ and $t = t_{k+1}$,

$$\mathbb{E}\left(L_{t_{k+1}}^0(\overline{X}) - L_{t_k}^0(\overline{X})\Big/\left\{\overline{X}_{t_k} = x\right\}\right) = \sigma\int_0^{\Delta t}\frac{\sqrt{x}}{\sqrt{2\pi s}}\exp\left(-\frac{(x+b(x)s)^2}{2\sigma^2 x\ s}\right)ds.$$

From (H1), $b(x) \geq -Kx$, with $K \geq 0$. Then,

$$\mathbb{E}\left(L_{t_{k+1}}^0(\overline{X}) - L_{t_k}^0(\overline{X})\Big/\mathcal{F}_{t_k}\right) \leq \sigma\int_0^{\Delta t}\frac{\sqrt{\overline{X}_{t_k}}}{\sqrt{2\pi s}}\exp\left(-\frac{\overline{X}_{t_k}(1-Ks)^2}{2\sigma^2 s}\right)ds.$$

For Δt sufficiently small, $1 - K\Delta t \geq 1/2$ and

$$\mathbb{E}\left(L_{t_{k+1}}^0(\overline{X}) - L_{t_k}^0(\overline{X})\Big/\mathcal{F}_{t_k}\right) \leq \sigma\int_0^{\Delta t}\frac{\sqrt{\overline{X}_{t_k}}}{\sqrt{2\pi s}}\exp\left(-\frac{\overline{X}_{t_k}}{8\sigma^2\Delta t}\right)ds.$$

Now we use the upper-bound $a\exp(-\frac{a^2}{2}) \leq 1, \forall a \in \mathbb{R}$, to get

$$\begin{aligned}\mathbb{E}\left(L_{t_{k+1}}^0(\overline{X}) - L_{t_k}^0(\overline{X})\right) &\leq \sigma^2\int_0^{\Delta t}\frac{2\sqrt{\Delta t}}{\sqrt{\pi s}}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_{t_k}}{16\sigma^2\Delta t}\right)\right]ds \\ &\leq \frac{4\sigma^2\Delta t}{\sqrt{\pi}}\sup_{k=0,\ldots,N}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{16\sigma^2\Delta t}\right).\end{aligned}$$

We sum over k and apply the Lemma 3.6 to end the proof. ☐

3.2 Proof of Theorem 2.3

We are now in position to prove the Theorem 2.3. To study the weak error $\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T)$, we use the Feynman–Kac representation of the solution $u(t,x)$ to the Cauchy problem (17), studied in the Proposition 3.2: for all $(t,x) \in [0,T] \times (0,+\infty)$, $\mathbb{E}f(X^x_{T-t}) = u(t,x)$. Thus the weak error becomes

$$\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T) = \mathbb{E}\left(u(0,x_0) - u(T,\overline{X}_T)\right)$$

with $(\overline{X})$ satisfying (27). Applying the Itô's formula a first time, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[u(0,x_0) - u(T,\overline{X}_T)\right] \\
&= -\int_0^T \mathbb{E}\left[\frac{\partial u}{\partial t}(s,\overline{X}_s) + \operatorname{sgn}(\overline{Z}_s) b(\overline{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\overline{X}_s) + \frac{\sigma^2}{2}\overline{X}_{\eta(s)}\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_s)\right] ds \\
&\quad - \mathbb{E}\int_0^T \operatorname{sgn}(\overline{Z}_s)\sigma\sqrt{\overline{X}_{\eta(s)}}\frac{\partial u}{\partial x}(s,\overline{X}_s) dW_s - \mathbb{E}\int_0^T \frac{1}{2}\frac{\partial u}{\partial x}(s,\overline{X}_s) dL^0(\overline{X})_s.
\end{aligned}$$

From Proposition 3.2 and Lemma 2.1, we easily check that the stochastic integral $(\int_0^\cdot \operatorname{sgn}(\overline{Z}_s)\sqrt{\overline{X}_{\eta(s)}}\frac{\partial u}{\partial x}(s,\overline{X}_s)dW_s)$ is a martingale. Furthermore, we use the Cauchy problem (17) to get

$$\begin{aligned}
&\mathbb{E}\left[u(0,x_0) - u(T,\overline{X}_T)\right] \\
&= -\int_0^T \mathbb{E}\left[\left(b(\overline{X}_{\eta(s)}) - b(\overline{X}_s)\right)\frac{\partial u}{\partial x}(s,\overline{X}_s) + \frac{\sigma^2}{2}\left(\overline{X}_{\eta(s)} - \overline{X}_s\right)\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_s)\right] ds \\
&\quad - \mathbb{E}\int_0^T \frac{1}{2}\frac{\partial u}{\partial x}(s,\overline{X}_s) dL^0(\overline{X})_s + \int_0^T 2\mathbb{E}\left(\mathbb{1}_{\{\overline{Z}_s \le 0\}} b(\overline{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\overline{X}_s)\right) ds.
\end{aligned}$$

From Proposition 3.5,

$$\left|\mathbb{E}\int_0^T \frac{\partial u}{\partial x}(s,\overline{X}_s) dL^0(\overline{X})_s\right| \le \left\|\frac{\partial u}{\partial x}\right\|_\infty \mathbb{E}\left(L^0_T(\overline{X})\right) \le C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

On the other hand, by Lemma 3.7 for any $s \in [0,T]$,

$$\left|2\mathbb{E}\left(\mathbb{1}_{\{\overline{Z}_s \le 0\}} b(\overline{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\overline{X}_s)\right)\right| \le \left\|\frac{\partial u}{\partial x}\right\|_\infty \mathbb{E}\left[\left(b(0) + K\overline{X}_{\eta(s)}\right)\exp\left(-\frac{\overline{X}_{\eta(s)}}{8\sigma^2\Delta t}\right)\right].$$

As for any $x \ge 0$, $x\exp(-\frac{x}{16\sigma^2\Delta t}) \le 16\sigma^2\Delta t$, we conclude, by Lemma 3.6, that

$$\left|\int_0^T 2\mathbb{E}\left(\mathbb{1}_{\{\overline{Z}_s \le 0\}} b(\overline{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\overline{X}_s)\right) ds\right| \le C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

Hence,

$$\begin{aligned}
&\left|\mathbb{E}\left[u(0,x_0) - u(T,\overline{X}_T)\right]\right| \\
&\le \left|\int_0^T \mathbb{E}\left[\left(b(\overline{X}_{\eta(s)}) - b(\overline{X}_s)\right)\frac{\partial u}{\partial x}(s,\overline{X}_s) + \frac{\sigma^2}{2}\left(\overline{X}_{\eta(s)} - \overline{X}_s\right)\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_s)\right] ds\right| + C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.
\end{aligned}$$

By applying the Itô's formula a second time ($\frac{\partial u}{\partial x}$ is a C^3 function with bounded derivatives),

$$\begin{aligned}
&\mathbb{E}\left[\left(b(\overline{X}_s)-b(\overline{X}_{\eta(s)}\right)\frac{\partial u}{\partial x}(s,\overline{X}_s)\right]\\
&=\mathbb{E}\int_{\eta(s)}^{s}\operatorname{sgn}(\overline{Z}_\theta)b(\overline{X}_{\eta(s)})\left[\left(b(\overline{X}_\theta)-b(\overline{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_\theta)+b'(\overline{X}_\theta)\frac{\partial u}{\partial x}(s,\overline{X}_\theta)\right]d\theta\\
&+\mathbb{E}\int_{\eta(s)}^{s}\frac{\sigma^2}{2}\overline{X}_{\eta(s)}\left[\left(b(\overline{X}_\theta)-b(\overline{X}_{\eta(s)}\right)\frac{\partial^3 u}{\partial x^3}(s,\overline{X}_\theta)+2b'(\overline{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_\theta)+b''(\overline{X}_\theta)\frac{\partial u}{\partial x}(s,\overline{X}_\theta)\right]d\theta\\
&+\mathbb{E}\int_{\eta(s)}^{s}\frac{1}{2}\left[\left(b(0)-b(\overline{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,0)+b'(0)\frac{\partial u}{\partial x}(s,0)\right]dL^0_\theta(\overline{X})
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(b(\overline{X}_s)-b(\overline{X}_{\eta(s)}\right)\frac{\partial u}{\partial x}(s,\overline{X}_s)\right]\right|\\
&\leq C\Delta t\left(1+\sup_{0\leq\theta\leq T}\mathbb{E}|\overline{X}_\theta|^2+\mathbb{E}\left\{(1+|\overline{X}_{\eta(s)}|)\left(L^0_s(\overline{X})-L^0_{\eta(s)}(\overline{X})\right)\right\}\right)
\end{aligned}$$

and we conclude by Lemma 2.1 and Proposition 3.5 that

$$\left|\mathbb{E}\left[\left(b(\overline{X}_s)-b(\overline{X}_{\eta(s)}\right)\frac{\partial u}{\partial x}(s,\overline{X}_s)\right]\right|\leq C\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right).$$

By similar arguments, we show that

$$\left|\mathbb{E}\left[\left(\overline{X}_s-\overline{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\overline{X}_s)\right]\right|\leq C\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right)$$

which ends the proof of Theorem 2.3.

4 The case of processes with $1/2<\alpha<1$

4.1 Preliminary results

In this section, (X_t) denotes the solution of (11) starting at x_0 at time 0 and (X^x_t), starting at $x\geq 0$ at time 0, is the unique strong solution to

$$X^x_t=x+\int_0^t b(X^x_s)ds+\sigma\int_0^t (X^x_s)^\alpha\,dW_s. \tag{35}$$

4.1.1 On the exact solution

We give some upper-bounds on inverse moments and exponential moments of (X_t).

Lemma 4.1. *Assume* (H1). *Let $x>0$. For any $1/2<\alpha<1$, for any $p>0$, there exists a positive constant C, depending on the parameters of the model* (35) *and on p such that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[(X^x_t)^{-p}\right]\leq C(1+x^{-p}).$$

Proof. Let τ_n be the stopping time defined by $\tau_n = \inf\{0 < s \leq T; X_s^x \leq 1/n\}$. By the Itô's formula,

$$\begin{aligned}
\mathbb{E}\left[(X^x_{t\wedge\tau_n})^{-p}\right] =& x^{-p} - p\mathbb{E}\left[\int_0^{t\wedge\tau_n} \frac{b(X_s^x)ds}{(X_s^x)^{p+1}}\right] + p(p+1)\frac{\sigma^2}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_n} \frac{ds}{(X_s^x)^{p+2(1-\alpha)}}\right] \\
\leq& x^{-p} + pK\int_0^t \mathbb{E}\left(\frac{1}{(X^x_{s\wedge\tau_n})^p}\right)ds \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(p(p+1)\frac{\sigma^2}{2}\frac{1}{(X_s^x)^{p+2(1-\alpha)}} - p\frac{b(0)}{(X_s^x)^{p+1}}\right)ds\right].
\end{aligned}$$

It is possible to find a positive constant C such that, for any $x > 0$,

$$\left(p(p+1)\frac{\sigma^2}{2}\frac{1}{x^{p+2(1-\alpha)}} - p\frac{b(0)}{x^{p+1}}\right) \leq C.$$

An easy computation shows that $\underline{C} = p\,(2\alpha-1)\,\frac{\sigma^2}{2}\left[(p+2(1-\alpha))\frac{\sigma^2}{2b(0)}\right]^{\frac{p+2(1-\alpha)}{2\alpha-1}}$ is the smallest one satisfying the upper-bound above. Hence,

$$\mathbb{E}\left[(X^x_{t\wedge\tau_n})^{-p}\right] \leq x^{-p} + \underline{C}T + pK\int_0^t \sup_{\theta\in[0,s]}\mathbb{E}\left[(X^x_{\theta\wedge\tau_n})^{-p}\right]ds$$

and by the Gronwall Lemma

$$\sup_{t\in[0,T]}\mathbb{E}\left[(X^x_{t\wedge\tau_n})^{-p}\right] \leq \left(x^{-p} + \underline{C}T\right)\exp(pKT).$$

We end the proof, by taking the limit $n \to +\infty$. □

Lemma 4.2. *Assume* (H1).

(i) For any $a \geq 0$, for all $0 \leq t \leq T$, a.s. $(X_t^x)^{2(1-\alpha)} \geq r_t(a)$, where $(r_t(a), 0 \leq t \leq T)$ is the solution of the CIR Equation:

$$r_t(a) = x^{2(1-\alpha)} + \int_0^t (a - \lambda(a)r_s(a))ds + 2\sigma(1-\alpha)\int_0^t \sqrt{r_s(a)}dW_s$$

with

$$\lambda(a) = 2(1-\alpha)K + \left(\frac{(2\alpha-1)^{2\alpha-1}\left(a + \sigma^2(1-\alpha)(2\alpha-1)\right)}{b(0)^{2(1-\alpha)}}\right)^{\frac{1}{2\alpha-1}}. \tag{36}$$

(ii) For all $\mu \geq 0$, there exists a constant $C(T,\mu) > 0$ with a non decreasing dependency on T and μ, depending also on K, $b(0)$, σ, α and x such that

$$\mathbb{E}\exp\left(\mu\int_0^T \frac{ds}{(X_s^x)^{2(1-\alpha)}}\right) \leq C(T,\mu). \tag{37}$$

(iii) The process $(M_t^x, 0 \leq t \leq T)$ defined by

$$M_t^x = \exp\left(\alpha\sigma\int_0^t \frac{dW_s}{(X_s^x)^{1-\alpha}} - \alpha^2\frac{\sigma^2}{2}\int_0^t \frac{ds}{(X_s^x)^{2(1-\alpha)}}\right) \tag{38}$$

is a martingale. Moreover for all $p \geq 1$, there exists a positive constant $C(T,p)$ depending also on $b(0)$, σ and α, such that

$$\mathbb{E}\left(\sup_{t\in[0,\,T]}(M_t^x)^p\right) \leq C(T,p)\left(1+\frac{1}{x^{\alpha p}}\right). \tag{39}$$

Proof. Let $Z_t = (X_t^x)^{2(1-\alpha)}$. By the Itô's formula,

$$Z_t = x^{2(1-\alpha)} + \int_0^t \beta(Z_s)ds + 2(1-\alpha)\sigma\int_0^t \sqrt{Z_s}dW_s,$$

where, for all $x > 0$, the drift coefficient $\beta(x)$ is defined by

$$\beta(x) = 2(1-\alpha)b(x^{\frac{1}{2(1-\alpha)}})x^{-\frac{2\alpha-1}{2(1-\alpha)}} - \sigma^2(1-\alpha)(2\alpha-1).$$

From (H1), $b(x) \geq b(0) - Kx$ and for all $x > 0$, $\beta(x) \geq \bar{\beta}(x)$, where we set

$$\bar{\beta}(x) = 2(1-\alpha)b(0)x^{-\frac{2\alpha-1}{2(1-\alpha)}} - 2(1-\alpha)Kx - \sigma^2(1-\alpha)(2\alpha-1).$$

For all $a \geq 0$ and $\lambda(a)$ given by (36), we consider $f(x) = \bar{\beta}(x) - a + \lambda(a)x$. An easy computation shows that $f(x)$ has one minimum at the point $x^\star = \left(\frac{b(0)(2\alpha-1)}{\lambda(a)-2(1-\alpha)K}\right)^{2(1-\alpha)}$. Moreover,

$$f(x^\star) = \frac{b(0)^{2(1-\alpha)}}{(2\alpha-1)^{2\alpha-1}}\left(\lambda(a) - 2(1-\alpha)K\right)^{2\alpha-1} - \left(a + \sigma^2(1-\alpha)(2\alpha-1)\right)$$

and when $\lambda(a)$ is given by (36), $f(x^\star) = 0$. We conclude that $\beta(x) \geq a - \lambda(a)x$ and (i) holds by the Comparison Theorem for the solutions of one-dimensional SDE. As a consequence,

$$\mathbb{E}\exp\left(\mu\int_0^T \frac{ds}{(X_s^x)^{2(1-\alpha)}}\right) \leq \mathbb{E}\left(\exp\left(\mu\int_0^T \frac{ds}{r_s(a)}\right)\right).$$

We want to apply the Lemma A.2, on the exponential moments of the CIR process. To this end, we must choose the constant a such that $a \geq 4(1-\alpha)^2\sigma^2$ and $\mu \leq \frac{\nu^2(a)(1-\alpha)^2 4\sigma^2}{8}$, for $\nu(a)$ as in Lemma A.2. An easy computation shows that $a = 4(1-\alpha)^2\sigma^2 \vee \left(2(1-\alpha)^2\sigma^2 + (1-\alpha)\sigma 2\sqrt{2\mu}\right)$ is convenient and (ii) follows by applying the Lemma A.2 to the process $(r_t(a), 0 \leq t \leq T)$. Thanks to (ii), the Novikov criteria applied to M_t^x is clearly satisfied. Moreover, by the integration by parts formula.

$$\begin{aligned} M_t^x &= \left(\frac{X_t^x}{x}\right)^\alpha \exp\left(\int_0^t\left(-\alpha\frac{b(X_s^x)}{X_s^x} + \alpha(1-\alpha)\frac{\sigma^2}{2}\frac{1}{(X_s^x)^{2(1-\alpha)}}\right)ds\right) \\ &\leq \left(\frac{X_t^x}{x}\right)^\alpha \exp(KT)\exp\left(\int_0^t\left(-\alpha\frac{b(0)}{X_s^x} + \alpha(1-\alpha)\frac{\sigma^2}{2}\frac{1}{(X_s^x)^{2(1-\alpha)}}\right)ds\right). \end{aligned}$$

To end the proof, notice that it is possible to find a positive constant λ such that, for any $x > 0$, $-\frac{b(0)\alpha}{x} + \frac{\sigma^2\alpha(1-\alpha)}{2}\frac{1}{x^{2(1-\alpha)}} \leq \lambda$. An easy computation shows that

$$\underline{\lambda} = \frac{\alpha}{2}(2\alpha-1)\left[\frac{(1-\alpha)^{3-2\alpha}\sigma^2}{b(0)^{2(1-\alpha)}}\right]^{\frac{1}{2\alpha-1}}.$$

is convenient. Thus, $M_t^x \leq \left(\frac{X_t^x}{x}\right)^\alpha \exp\left((K+\underline{\lambda})T\right)$ and we conclude by using the Lemma 2.1. ☐

4.1.2 On the associated Kolmogorov PDE

Proposition 4.3. *Let $1/2 < \alpha \leq 1$. Let f be a $\mathbb{R}$-valued C^4 bounded function, with bounded spatial derivatives up to the order 4. We consider the $\mathbb{R}$-valued function defined on $[0,T] \times [0,+\infty)$ by $u(t,x) = \mathbb{E} f(X^x_{T-t})$. Then under* (H1) *and* (H2)*, u is in $C^{1,4}([0,T] \times (0,+\infty))$ and there exists a positive constant C depending on f, b and T such that*

$$\|u\|_{L^\infty([0,T]\times[0,+\infty))} + \left\|\frac{\partial u}{\partial x}\right\|_{L^\infty([0,T]\times[0,+\infty))} \leq C$$

and for all $x > 0$,

$$\sup_{t\in[0,T]} \left|\frac{\partial u}{\partial t}(t,x)\right| \leq C(1+x^{2\alpha}),$$

$$\text{and} \quad \sup_{t\in[0,T]} \sum_{k=2}^{4} \left|\frac{\partial^k u}{\partial x^k}\right|(t,x) \leq C\left(1+\frac{1}{x^{q(\alpha)}}\right),$$

where the constant $q(\alpha) > 0$ depends only on α. Moreover, $u(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + b(x)\dfrac{\partial u}{\partial x}(t,x) + \dfrac{\sigma^2}{2} x^{2\alpha} \dfrac{\partial^2 u}{\partial x^2}(t,x) = 0, \ (t,x) \in [0,T] \times (0,+\infty), \\ u(T,x) = f(x), \ x \in [0,+\infty). \end{cases} \tag{40}$$

The following Proposition 4.4 allows us to compute the derivatives of $u(t,x)$. Equation (35) has locally Lipschitz coefficients on $(0,+\infty)$, with locally Lipschitz first order derivatives. Then X^x_t is continuously differentiable and if we denote $J^x_t = \frac{dX^x_t}{dx}$, the process $(J^x_t, 0 \leq t \leq T)$ satisfies the linear equation

$$J^x_t = 1 + \int_0^t J^x_s b'(X^x_s) ds + \int_0^t \alpha\sigma J^x_s \frac{dW_s}{(X^x_s)^{1-\alpha}}. \tag{41}$$

Proposition 4.4. *Assume* (H1) *and* (H2)*. Let $g(x)$, $h(x)$ and $k(x)$ be some C^1 functions on $(0,+\infty)$ such that, there exist $p_1 > 0$ and $p_2 > 0$,*

$$\forall x > 0, \quad \begin{array}{l} |g(x)| + |g'(x)| + |h(x)| + |h'(x)| + |k'(x)| \leq C\left(1 + x^{p_1} + \frac{1}{x^{p_2}}\right), \\ |k(x)| \leq C\left(1 + \frac{1}{x^{2(1-\alpha)}}\right). \end{array}$$

Let v be the $\mathbb{R}$-valued function defined on $[0,T] \times (0,+\infty)$ by

$$v(t,x) = \mathbb{E}\left[g(X^x_t)\exp(\int_0^t k(X^x_s)ds)\right] + \int_0^t \mathbb{E}\left[h(X^x_s)\exp(\int_0^s k(X^x_\theta)d\theta)\right] ds.$$

Then $v(t,x)$ is of class C^1 with respect to x and

$$\begin{aligned} \frac{\partial v}{\partial x}(t,x) =& \mathbb{E}\left[\exp(\int_0^t k(X^x_s)ds)\left(g'(X^x_t)J^x_t + g(X^x_t)\int_0^t k'(X^x_s)J^x_s ds\right)\right] \\ &+ \int_0^t \mathbb{E}\left[\exp(\int_0^s k(X^x_\theta)d\theta)\left(h'(X^x_s)J^x_s + h(X^x_s)\int_0^s k'(X^x_\theta)J^x_\theta d\theta\right)\right] ds. \end{aligned}$$

The proof is postponed in the Appendix B.

Proof of Proposition 4.3. Many arguments are similar to those of the proof of Proposition 3.2. Here, we restrict our attention on the main difficulty which consists in obtaining the upper bounds for the spatial derivatives of $u(t,x)$ up to

the order 4. By Lemma 4.1, for $x > 0$, $(\int_0^t \frac{dW_s}{(X_s^x)^{1-\alpha}}, 0 \leq t \leq T)$ is a locally square integrable martingale. Then J_t^x is given by

$$J_t^x = \exp\left(\int_0^t b'(X_s^x)ds + \alpha\sigma \int_0^t \frac{dW_s}{(X_s^x)^{1-\alpha}} - \frac{\sigma^2\alpha^2}{2} \int_0^t \frac{ds}{(X_s^x)^{2(1-\alpha)}}\right)$$

Or equivalently $J_t^x = \exp\left(\int_0^t b'(X_s^x)ds\right) M_t$, where (M_t) is the martingale defined in (38) and satisfying (39). Thus , we have $J_t^x = \exp\left(\int_0^t b'(X_s^x)ds\right) M_t$. b' being bounded, $\mathbb{E}J_t^x \leq \exp(KT)$ and for all $p > 1$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} (J_t^x)^p\right) \leq C(T)\left(1 + \frac{1}{x^{\alpha p}}\right). \tag{42}$$

By Proposition 4.4, $u(t,x)$ is differentiable and

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}\left[f'(X_{T-t}^x)J_{T-t}^x\right].$$

Then, $|\frac{\partial u}{\partial x}(t,x)| \leq \|f'\|_\infty \exp(KT)$. The integration by parts formula gives

$$J_t^x = \frac{(X_t^x)^\alpha}{x^\alpha} \exp\left(\int_0^t \left(b'(X_s^x) - \frac{\alpha b(X_s^x)}{X_s^x} + \frac{\sigma^2\alpha(1-\alpha)}{2(X_s^x)^{2(1-\alpha)}}\right) ds\right).$$

We apply again the Proposition 4.4 to compute $\frac{\partial^2 u}{\partial x^2}(t,x)$: for any $x > 0$,

$$\frac{dJ_t^x}{dx} = -\frac{\alpha J_t^x}{x} + \frac{\alpha (J_t^x)^2}{X_t^x} + J_t^x\left(\int_0^t \left(b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} - \frac{\sigma^2\alpha(1-\alpha)^2}{(X_s^x)^{3-2\alpha}}\right) J_s^x ds\right)$$

and

$$\begin{aligned}
&\tfrac{\partial^2 u}{\partial x^2}(t,x) = \mathbb{E}\left[f''(X_{T-t}^x)(J_{T-t}^x)^2\right] - \frac{\alpha}{x}\frac{\partial u}{\partial x}(t,x) + \alpha\mathbb{E}\left[\frac{(J_{T-t}^x)^2}{X_{T-t}^x} f'(X_{T-t}^x)\right] \\
&+\mathbb{E}\left[f'(X_{T-t}^x)J_{T-t}^x \int_0^{T-t} \left(b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} - \frac{\sigma^2\alpha(1-\alpha)^2}{(X_s^x)^{3-2\alpha}}\right) J_s^x ds\right].
\end{aligned} \tag{43}$$

By using the Cauchy-Schwarz Inequality with Lemma 4.1 and estimate (42), the second term on the right-hand side is bounded by

$$\begin{aligned}
&\|f'\|_\infty \mathbb{E}\left[\sup_{t\in[0,T]} (J_t^x)^2 \int_0^{T-t} \left|b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} - \frac{\sigma^2\alpha(1-\alpha)^2}{(X_s^x)^{3-2\alpha}}\right| ds\right] \\
&\leq C(T)\left(1 + \frac{1}{x^{2(1+\alpha)}}\right).
\end{aligned}$$

By using similar arguments, it comes that

$$\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| \leq C(T)\left(1 + \frac{1}{x^{2+2\alpha}}\right).$$

We apply again the Proposition 4.4 to compute $\frac{\partial^3 u}{\partial x^3}(t,x)$ from (43) and next $\frac{\partial^4 u}{\partial x^4}(t,x)$, the main difficulty being the number of terms to write. In view of the expression of $\frac{dJ_s^x}{dx}$, each term can be bounded by $C(T)(1 + x^{-2(n-1)-n\alpha})$, where n is the derivation order, by using the Cauchy-Schwarz Inequality and the upper-bounds $\mathbb{E} \sup_{t\in[0,T]} (J_t^x)^p \leq C(T)(1 + x^{-\alpha p})$ and $\sup_{t\in[0,T]} \mathbb{E}(X_t^x)^{-p} \leq C(1 + x^{-p})$. $\square$

4.1.3 On the approximation process

When $1/2 < \alpha < 1$, according to (3) and (6), the discrete time process $(\overline{X})$ associated to (X) is

$$\begin{cases} \overline{X}_0 = x_0, \\ \overline{X}_{t_{k+1}} = \left| \overline{X}_{t_k} + b(\overline{X}_{t_k})\Delta t + \sigma \overline{X}_{t_k}^{\alpha}(W_{t_{k+1}} - W_{t_k}) \right|, \ k = 0, \ldots, N-1, \end{cases}$$

Its time continuous version $(\overline{X}_t, 0 \leq t \leq T)$ satisfies

$$\overline{X}_t = x_0 + \int_0^t \operatorname{sgn}(\overline{Z}_s) b(\overline{X}_{\eta(s)}) ds + \sigma \int_0^t \operatorname{sgn}(\overline{Z}_s) \overline{X}_{\eta(s)}^{\alpha} dW_s + \frac{1}{2} L_t^0(\overline{X}), \tag{44}$$

where for any $t \in [0, T]$, we set

$$\overline{Z}_t = \overline{X}_{\eta(t)} + (t - \eta(t)) b(\overline{X}_{\eta(t)}) + \sigma \overline{X}_{\eta(t)}^{\alpha} (W_t - W_{\eta(t)}), \tag{45}$$

so that, for all $t \in [0,\, T]$, $\overline{X}_t = |\overline{Z}_t|$.

In the sequel, we will use the following notation:

$$\mathcal{O}_{\exp}(\Delta t) = C(T) \exp\left(-\frac{C}{\Delta t^{\alpha - \frac{1}{2}}} \right),$$

where the positive constants C and $C(T)$ are independent of Δt but can depend on α, σ and $b(0)$. $C(T)$ is non-decreasing in T. The quantity $\mathcal{O}_{\exp}(\Delta t)$ decreases exponentially fast with Δt.

In this section, we are interested in the behavior of the processes $(\overline{X})$ or $(\overline{Z})$ near 0. We work under the hypothesis (H3'): $x_0 > \frac{b(0)}{\sqrt{2}} \Delta t$. We introduce the stopping time τ defined by

$$\tau = \inf \left\{ s \geq 0; \overline{X}_s < \frac{b(0)}{2} \Delta t \right\}. \tag{46}$$

Under (H3'), we are able to control probabilities like $\mathbb{P}(\tau \leq T)$. This is an important difference with the case $\alpha = 1/2$.

Lemma 4.5. *Assume* (H1), (H2) *and* (H3'). *Then*

$$\mathbb{P}\left(\tau \leq T\right) \leq \mathcal{O}_{\exp}(\Delta t). \tag{47}$$

Proof. The first step of the proof consists in obtaining the following estimate:

$$\forall k \in \{0, \ldots, N\},\ \mathbb{P}\left(\overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}} \Delta t \right) \leq \mathcal{O}_{\exp}(\Delta t). \tag{48}$$

Indeed, as $b(x) \geq b(0) - Kx$ for $x \geq 0$, for $k \geq 1$,

$$\begin{aligned} &\mathbb{P}\left(\overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}} \Delta t \right) \\ &\leq \mathbb{P}\left(W_{t_k} - W_{t_{k-1}} \leq \frac{-\overline{X}_{t_{k-1}}(1 - K\Delta t) - b(0)(1 - \frac{1}{\sqrt{2}})\Delta t}{\sigma \overline{X}_{t_{k-1}}^{\alpha}}, \overline{X}_{t_{k-1}} > 0 \right). \end{aligned}$$

As Δt is sufficiently small, by using the Gaussian inequality $\mathbb{P}(G \leq \beta) \leq 1/2 \exp(-\beta^2/2)$, for a standard Normal r.v. G and $\beta < 0$, we get

$$\begin{aligned} &\mathbb{P}\left(\overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}} \Delta t \right) \\ &\leq \mathbb{E}\left[\exp\left(-\frac{\left(\overline{X}_{t_{k-1}}(1 - K\Delta t) + b(0)(1 - \frac{1}{\sqrt{2}})\Delta t \right)^2}{2\sigma^2 \overline{X}_{t_{k-1}}^{2\alpha} \Delta t} \right) \mathbb{1}_{\{\overline{X}_{t_{k-1}} > 0\}} \right] \\ &\leq \mathbb{E}\left[\exp\left(-\frac{\overline{X}_{t_{k-1}}^{2(1-\alpha)}}{8\sigma^2 \Delta t} \right) \exp\left(-\frac{b(0)(1 - \frac{1}{\sqrt{2}})}{2\sigma^2 \overline{X}_{t_{k-1}}^{2\alpha - 1}} \right) \mathbb{1}_{\{\overline{X}_{t_{k-1}} > 0\}} \right]. \end{aligned}$$

By separating the events $\left\{\overline{X}_{t_{k-1}} \geq \sqrt{\Delta t}\right\}$ and $\left\{\overline{X}_{t_{k-1}} < \sqrt{\Delta t}\right\}$ in the expectation above, we obtain

$$\mathbb{P}\left(\overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \leq \exp\left(-\frac{1}{8\sigma^2 \Delta t^\alpha}\right) + \exp\left(-\frac{b(0)(1-\frac{1}{\sqrt{2}})}{2\sigma^2(\Delta t)^{\alpha-\frac{1}{2}}}\right) = \mathcal{O}_{\exp}(\Delta t).$$

Now we prove (47). Notice that

$$\mathbb{P}(\tau \leq T) \leq \sum_{k=0}^{N-1} \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \overline{Z}_s \leq \frac{b(0)}{2}\Delta t\ ,\ \overline{X}_{t_k} > \frac{b(0)}{2}\Delta t\right).$$

For each $k \in \{0, 1, \ldots, N-1\}$, by using (48) and $b(x) \leq b(0) - Kx$, we have

$$\begin{aligned}
&\mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \overline{Z}_s \leq \frac{b(0)}{2}\Delta t,\ \overline{X}_{t_k} > \frac{b(0)}{2}\Delta t\right) \\
&= \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \overline{Z}_s \leq \frac{b(0)}{2}\Delta t,\ \overline{X}_{t_k} > \frac{b(0)}{2}\Delta t,\ \overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \\
&\quad + \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \overline{Z}_s \leq \frac{b(0)}{2}\Delta t,\ \overline{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right) \\
&\leq \mathbb{P}\left(\overline{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) + \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \overline{Z}_s \leq \frac{b(0)}{2}\Delta t, \overline{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right) \\
&\leq \mathcal{O}_{\exp}(\Delta t) + \mathbb{E}\left\{\mathbb{1}_{\left(\overline{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right)} \mathbb{P}\left(\inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{b(0) - Kx}{\sigma x^\alpha} s + B_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha}\right)\Big|_{x = \overline{X}_{t_k}}\right\},
\end{aligned}$$

where (B_t) denotes a Brownian motion independent of (W_t). The proof is ended if we show that

$$\mathbb{P}\left(\inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{(b(0) - Kx)}{\sigma x^\alpha} s + B_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha}\right) \leq \mathcal{O}_{\exp}(\Delta t), \text{ for } x \geq \frac{b(0)}{\sqrt{2}}\Delta t.$$

We use the following formula (see [2]): if $(B_t^\mu, 0 \leq t \leq T)$ denotes a Brownian motion with drift μ, starting at y_0, then for all $y \leq y_0$,

$$\mathbb{P}\left(\inf_{0 < s < t} B_s^\mu \leq y\right) = \frac{1}{2}\text{erfc}\left(\frac{y_0 - y}{\sqrt{2t}} + \frac{\mu\sqrt{t}}{\sqrt{2}}\right) + \frac{1}{2}\exp(2\mu(y - y_0))\text{erfc}\left(\frac{y_0 - y}{\sqrt{2t}} - \frac{\mu\sqrt{t}}{\sqrt{2}}\right),$$

where $\text{erfc}(z) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{\sqrt{2}z}^\infty \exp\left(-\frac{y^2}{2}\right) dy$, for all $z \in \mathbb{R}$. We set $\mu = \frac{(b(0) - Kx)}{\sigma x^\alpha}$, $y_0 = \frac{x^{1-\alpha}}{\sigma}$ and we choose $y = \frac{b(0)\Delta t}{2\sigma x^\alpha}$ satisfying $y \leq y_0$, if $x > \frac{b(0)}{\sqrt{2}}\Delta t$. Then

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{(b(0) - Kx)}{\sigma x^\alpha} s + W_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha}\right) \\
&= \frac{1}{2}\text{erfc}\left(\frac{x - \frac{b(0)\Delta t}{2}}{\sigma x^\alpha \sqrt{2\Delta t}} + \frac{(b(0) - Kx)\sqrt{\Delta t}}{\sigma x^\alpha \sqrt{2}}\right) \\
&\quad + \frac{1}{2}\exp\left(-\frac{2(b(0) - Kx)}{\sigma^2 x^{2\alpha}}\left(x - \frac{b(0)\Delta t}{2}\right)\right)\text{erfc}\left(\frac{x - \frac{b(0)\Delta t}{2}}{\sigma x^\alpha \sqrt{2\Delta t}} - \frac{(b(0) - Kx)\sqrt{\Delta t}}{\sigma x^\alpha \sqrt{2}}\right) \\
&:= A(x) + B(x).
\end{aligned}$$

For any $z \geq 0$, $\text{erfc}(z) \leq \exp(-z^2)$. Then, if Δt is sufficiently small, we have

$$A(x) \leq \exp\left(-\frac{(x(1 - K\Delta t) + \frac{b(0)}{2}\Delta t)^2}{2\sigma^2 x^{2\alpha}\Delta t}\right) \leq \exp\left(-\frac{x^{2(1-\alpha)}}{8\sigma^2\Delta t}\right)$$

and, for $x \geq \frac{b(0)}{\sqrt{2}}\Delta t$, $A(x) \leq \exp\left(-\frac{2^{\alpha} b(0)^{2(1-\alpha)}}{16\sigma^2\ \Delta t^{2\alpha-1}}\right) = \mathcal{O}_{\exp}(\Delta t)$. Now we consider $B(x)$. If $x \geq \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$, then as for $A(x)$, we have

$$
\begin{aligned}
B(x) &\leq \exp\left(-\frac{2(b(0)-Kx)}{\sigma^2 x^{2\alpha}}\left(x-\frac{b(0)\Delta t}{2}\right)\right)\exp\left(-\frac{(x-\frac{b(0)\Delta t}{2}-(b(0)-Kx)\Delta t)^2}{\sigma^2 x^{2\alpha}2\Delta t}\right)\\
&= \exp\left(-\frac{(x(1-K\Delta t)+\frac{b(0)\Delta t}{2})^2}{\sigma^2 x^{2\alpha}2\Delta t}\right) \leq \exp\left(-\frac{x^{2(1-\alpha)}}{8\sigma^2\Delta t}\right)
\end{aligned}
$$

and $B(x) \leq \exp\left(-\frac{2^{\alpha}\ b(0)^{2(1-\alpha)}}{16\sigma^2\ \Delta t^{2\alpha-1}}\right) = \mathcal{O}_{\exp}(\Delta t)$, for $x \geq \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$.
If $\frac{b(0)}{\sqrt{2}}\Delta t \leq x < \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$, then $\frac{2(b(0)-Kx)}{\sigma^2 x^{2\alpha}}(x-\frac{b(0)\Delta t}{2}) \geq \frac{b(0)^2\Delta t(\frac{1}{\sqrt{2}}-\frac{1}{2})}{\sigma^2 x^{2\alpha}}$ and

$$
B(x) \leq \exp\left(-\frac{2(b(0)-Kx)}{\sigma^2 x^{2\alpha}}\left(x-\frac{b(0)\Delta t}{2}\right)\right) \leq \exp\left(-\frac{b(0)^2\Delta t(\frac{1}{\sqrt{2}}-\frac{1}{2})}{\sigma^2 x^{2\alpha}}\right).
$$

For $x \geq \frac{b(0)}{\sqrt{2}}\Delta t$, we get $B(x) \leq \exp\left(-\frac{2^{2\alpha} b(0)^{2(1-\alpha)}(\frac{1}{\sqrt{2}}-\frac{1}{2})(1+K\Delta t)^{2\alpha}}{3^{2\alpha}\sigma^2(\Delta t)^{2\alpha-1}}\right) = \mathcal{O}_{\exp}(\Delta t)$. □

Lemma 4.6. *Assume* (H1), (H2) *and* (H3'). *Let τ be the stopping time defined in* (46). *For all $p \geq 0$, there exists a positive constant C depending on $b(0)$, σ, α, T and p but not on Δt, such that*

$$
\forall t \in [0,T],\ \ \mathbb{E}\left(\frac{1}{\overline{Z}^p_{t\wedge\tau}}\right) \leq C\left(1+\frac{1}{x_0^p}\right). \tag{49}
$$

Proof. First, we prove that

$$
\forall t \in [0,T],\quad \mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right) \leq \mathcal{O}_{\exp}(\Delta t). \tag{50}
$$

Indeed, while proceeding as in the proof of Lemma 4.5, we have

$$
\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right) \leq \mathbb{E}\exp\left(-\frac{\left(\overline{X}_{\eta(t)}(1-2K(t-\eta(t)))+2b(0)(t-\eta(t))\right)^2}{8\sigma^2(t-\eta(t))\overline{X}^{2\alpha}_{\eta(t)}}\right).
$$

By using $(a+b)^2 \geq a^2+2ab$, with $a = \overline{X}_{\eta(t)}(1-2K(t-\eta(t)))$ and $b = 2b(0)(t-\eta(t))$,

$$
\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right) \leq \mathbb{E}\left(\exp\left(-\frac{\overline{X}^{2(1-\alpha)}_{\eta(t)}(1-2K\Delta t)^2}{8\sigma^2\Delta t}\right)\exp\left(-\frac{b(0)(1-2K\Delta t)}{2\sigma^2\overline{X}^{2\alpha-1}_{\eta(t)}}\right)\right).
$$

For Δt sufficiently small,

$$
\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right) \leq \mathbb{E}\left(\exp\left(-\frac{\overline{X}^{2(1-\alpha)}_{\eta(t)}}{32\sigma^2\Delta t}\right)\exp\left(-\frac{b(0)}{4\sigma^2\overline{X}^{2\alpha-1}_{\eta(t)}}\right)\right).
$$

By separating the events $\left\{\overline{X}_{\eta(t)} \geq \sqrt{\Delta t}\right\}$ and $\left\{\overline{X}_{\eta(t)} < \sqrt{\Delta t}\right\}$ in the expectation above, we obtain

$$
\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right) \leq \exp\left(-\frac{1}{32\sigma^2(\Delta t)^{\alpha}}\right)+\exp\left(-\frac{b(0)}{4\sigma^2(\Delta t)^{\alpha-\frac{1}{2}}}\right) = \mathcal{O}_{\exp}(\Delta t).
$$

Now we prove (49). Notice that $\overline{Z}_{t\wedge\tau} = \overline{X}_{t\wedge\tau}$, by the Itô's formula

$$\frac{1}{\overline{Z}^p_{t\wedge\tau}} = \frac{1}{x_0^p} - p\int_0^{t\wedge\tau} \frac{b(\overline{X}_{\eta(s)})}{\overline{Z}_s^{p+1}}ds - p\sigma\int_0^{t\wedge\tau} \frac{\overline{X}^\alpha_{\eta(s)}}{\overline{Z}_s^{p+1}}dW_s + p(p+1)\frac{\sigma^2}{2}\int_0^{t\wedge\tau} \frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds.$$

Taking the expectation and using again $b(x) \geq b(0) - Kx$, we have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^p_{t\wedge\tau}}\right) \leq & \frac{1}{x_0^p} - p\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{b(0)}{\overline{Z}_s^{p+1}}ds\right) + pK\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{p+1}}ds\right) \\ & + p(p+1)\frac{\sigma^2}{2}\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right).\end{aligned}$$

By the definition of τ in (46),

$$\begin{aligned}\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{p+1}}ds\right) &= \mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s \leq \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{p+1}}ds\right) + \mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s > \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{p+1}}ds\right) \\ \leq \left(\frac{2}{b(0)\Delta t}\right)^{p+1} & T\sup_{t\in[0,T]}\left[\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right)\right]^{1/2}\sup_{t\in[0,T]}\left[\mathbb{E}\left(\overline{X}^2_{\eta(t)}\right)\right]^{1/2} + 2\int_0^t \mathbb{E}\left(\frac{1}{\overline{Z}^p_{s\wedge\tau}}\right)ds.\end{aligned}$$

We conclude, by the Lemma 2.1 and the upper-bound (50) that

$$\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{p+1}}ds\right) \leq C + 2\int_0^t \mathbb{E}\left(\frac{1}{\overline{Z}^p_{s\wedge\tau}}\right)ds.$$

Similarly,

$$\begin{aligned}\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right) &= \mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s \leq \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right) + \mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s > \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right) \\ &\leq \mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s \leq \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right) + 2^{2\alpha}\mathbb{E}\left(\int_0^{t\wedge\tau} \frac{ds}{\overline{Z}_s^{p+2(1-\alpha)}}\right).\end{aligned}$$

By using again the Lemma 2.1 and the upper-bound (50), we have

$$\begin{aligned}\mathbb{E}\left(\int_0^{t\wedge\tau} \mathbb{1}_{(\overline{Z}_s \leq \frac{\overline{X}_{\eta(s)}}{2})}\frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{p+2}}ds\right) &\leq T\left(\frac{2}{b(0)\Delta t}\right)^{p+2}\sup_{t\in[0,T]}\left[\sqrt{\mathbb{P}\left(\overline{Z}_t \leq \frac{\overline{X}_{\eta(t)}}{2}\right)}\sqrt{\mathbb{E}\left(\overline{X}^{4\alpha}_{\eta(t)}\right)}\right] \\ &\leq T\left(\frac{2}{b(0)\Delta t}\right)^{p+2}\mathcal{O}_{\exp}(\Delta t) \leq C.\end{aligned}$$

Finally,

$$\mathbb{E}\left(\frac{1}{\overline{Z}^p_{t\wedge\tau}}\right) \leq \frac{1}{x_0^p} + \mathbb{E}\int_0^{t\wedge\tau}\left(-\frac{pb(0)}{\overline{Z}_s^{p+1}} + \frac{2^{2\alpha-1}p(p+1)\sigma^2}{\overline{Z}_s^{p+2(1-\alpha)}}\right)ds + C\int_0^t \mathbb{E}\left(\frac{1}{\overline{Z}^p_{s\wedge\tau}}\right)ds + C.$$

We can easily check that there exists a positive constant C such that, for all $z > 0$, $\frac{-pb(0)}{z^{p+1}} + \frac{p(p+1)2^{2\alpha-1}\sigma^2}{z^{p+2(1-\alpha)}} \leq C$. Hence

$$\mathbb{E}\left(\frac{1}{\overline{Z}^p_{t\wedge\tau}}\right) \leq \frac{1}{x_0^p} + C\int_0^t \mathbb{E}\left(\frac{1}{\overline{Z}^p_{s\wedge\tau}}\right)ds + C$$

and we conclude by applying the Gronwall Lemma. □

4.2 Proof of Theorem 2.5

As in the proof of Theorem 2.3, we use the Feynman–Kac representation of the solution of the Cauchy problem (40), studied in the Proposition 4.3: for all $(t,x) \in [0,T] \times (0,+\infty)$, $\mathbb{E}f(X^x_{T-t}) = u(t,x)$. Thus, the weak error becomes

$$\mathbb{E}f(X_T) - \mathbb{E}f(\overline{X}_T) = \mathbb{E}\left(u(0,x_0) - u(T,\overline{X}_T)\right).$$

Let τ be the stopping time defined in (46). By Lemma 4.5,

$$\mathbb{E}\left(u(T\wedge\tau,\overline{Z}_{T\wedge\tau}) - u(T,\overline{X}_T)\right) \leq 2\left\|u\right\|_{L^\infty([0,T]\times[0,+\infty])}\mathbb{P}\left(\tau \leq T\right) \leq \mathcal{O}_{\exp}(\Delta t).$$

We bound the error by

$$\left|\mathbb{E}\left(u(T,\overline{X}_T) - u(0,x_0)\right)\right| \leq \left|\mathbb{E}\left(u(T\wedge\tau,\overline{Z}_{T\wedge\tau}) - u(0,x_0)\right)\right| + \mathcal{O}_{\exp}(\Delta t)$$

and we are now interested in $\mathbb{E}\left(u(T\wedge\tau,\overline{Z}_{T\wedge\tau}) - (u(0,x_0)\right)$. Let L and $\mathcal{L}_z$ the second order differential operators defined for any C^2 function $g(x)$ by

$$Lg(x) = b(x)\frac{\partial g}{\partial x}(x) + \frac{\sigma^2}{2}x^{2\alpha}\frac{\partial^2 g}{\partial x^2}(x) \quad \text{and} \quad \mathcal{L}_z g(x) = b(z)\frac{\partial g}{\partial x}(x) + \frac{\sigma^2}{2}z^{2\alpha}\frac{\partial^2 g}{\partial x^2}(x).$$

From Proposition 4.3, u is in $C^{1,4}([0,T]\times(0,+\infty))$ satisfying $\frac{\partial u}{\partial s}(t,x) + Lu(t,x) = 0$. $\overline{X}_t$ has bounded moments and the stopped process $(\overline{X}_{t\wedge\tau})$=$(\overline{Z}_{t\wedge\tau})$ has negative moments. Hence, applying the Itô's formula,

$$\mathbb{E}\left[u(T\wedge\tau,\,\overline{X}_{T\wedge\tau}) - u(0,x_0)\right] = \mathbb{E}\int_0^{T\wedge\tau}\left(\mathcal{L}_{\overline{X}_{\eta(s)}}u - Lu\right)(s,\overline{X}_s)ds,$$

Notice that

$$\begin{aligned}&\partial_\theta(\mathcal{L}_z u - Lu) + b(z)\partial_x(\mathcal{L}_z u - Lu) + \frac{\sigma^2}{2}z^{2\alpha}\partial^2_{x^2}(\mathcal{L}_z u - Lu)\\&= \mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2 u\end{aligned}$$

and by applying again the Itô's formula between $\eta(s)$ and s to $\left(\mathcal{L}_{\overline{X}_{\eta(s)}}u - Lu\right)(s,\overline{X}_s)$,

$$\begin{aligned}&\mathbb{E}\left[u(T\wedge\tau,\,\overline{X}_{T\wedge\tau}) - u(0,\,x_0)\right]\\&= \int_0^T\int_{\eta(s)}^s\mathbb{E}\left[\mathbb{1}_{(\theta\leq\tau)}\left(\mathcal{L}^2_{\overline{X}_{\eta(s)}}u - 2\mathcal{L}_{\overline{X}_{\eta(s)}}Lu + L^2u\right)(\theta,\overline{X}_\theta)\right]d\theta ds.\end{aligned}$$

$\left(\mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2u\right)(\theta,x)$ combines the derivatives of u up to the order four with b and its derivatives up to the order two and some power functions like the $z^{4\alpha}$ or $x^{2\alpha-2}$. When we value this expression at the point $(z,x) = (\overline{X}_{\eta(s\wedge\tau)},\overline{X}_{\theta\wedge\tau})$, with the upper bounds on the derivatives of u given in the Proposition 4.3 and the positive and negative moments of $\overline{X}$ given in the Lemmas 2.1 and 4.6, we get

$$\left|\mathbb{E}\left[\mathbb{1}_{(\theta\leq\tau)}\left(\mathcal{L}^2_{\overline{X}_{\eta(s)}}u - 2\mathcal{L}_{\overline{X}_{\eta(s)}}Lu + L^2u\right)(\theta,\overline{X}_\theta)\right]\right| \leq C\left(1 + \frac{1}{x_0^{q(\alpha)}}\right)$$

which implies the result of Theorem 2.5.

A On the Cox-Ingersoll-Ross model

In [6], Cox, Ingersoll and Ross proposed to model the dynamics of the short term interest rate as the solution of the following stochastic differential equation

$$\begin{cases} dr^x_t = (a - br^x_t)dt + \sigma\sqrt{r^x_t}dW_t,\\ r^x_0 = x \geq 0,\end{cases} \tag{51}$$

where $(W_t, 0 \leq t \leq T)$ is a one-dimensional Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a and σ are positive constants and $b \in \mathbb{R}$. For any $t \in [0,T]$, let $\mathcal{F}_t = \sigma(s \leq t, W_s)$.

Lemma A.1. *For any $x > 0$ and any $p > 0$,*

$$\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] = \frac{1}{\Gamma(p)}\left(\frac{2b}{\sigma^2(1-e^{-bt})}\right)^p \times \int_0^1 (2-\theta)\theta^{p-1}(1-\theta)^{\frac{2a}{\sigma^2}-p-1}\exp\left(-\frac{2bx\theta}{\sigma^2(e^{bt}-1)}\right)d\theta, \tag{52}$$

where $\Gamma(p) = \int_0^{+\infty} u^{p-1}\exp(-u)du$, $p > 0$, denotes the Gamma function. Moreover, if $a > \sigma^2$

$$\mathbb{E}\left[\frac{1}{r_t^x}\right] \leq \frac{e^{bt}}{x} \tag{53}$$

and, for any p such that $1 < p < \frac{2a}{\sigma^2} - 1$,

$$\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] \leq \frac{1}{\Gamma(p)}\left(\frac{2e^{|b|t}}{\sigma^2 t}\right)^p \text{ or } \mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] \leq C(p,T)\frac{1}{x^p}, \tag{54}$$

where $C(p,T)$ is a positive constant depending on p and T.

Proof. By the definition of the Gamma function, for all $x > 0$ and $p > 0$, $x^{-p} = \Gamma(p)^{-1}\int_0^{+\infty} u^{p-1}\exp(-ux)du$, so that

$$\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty} u^{p-1}\mathbb{E}\exp(-ur_t^x)du.$$

The Laplace transform of r_t^x is given by

$$\mathbb{E}\exp(-ur_t^x) = \frac{1}{(2uL(t)+1)^{2a/\sigma^2}}\exp\left(-\frac{uL(t)\zeta(t,x)}{2uL(t)+1}\right),$$

where $L(t) = \frac{\sigma^2}{4b}(1-\exp(-bt))$ and $\zeta(t,x) = \frac{4xb}{\sigma^2(\exp(bt)-1)} = xe^{-bt}/L(t)$, (see e.g. [15]). Hence,

$$\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty}\frac{u^{p-1}}{(2uL(t)+1)^{2a/\sigma^2}}\exp\left(-\frac{uL(t)\zeta(t,x)}{2uL(t)+1}\right)du.$$

By changing the variable $\theta = 2\frac{uL(t)}{2uL(t)+1}$ in the integral above, we obtain

$$\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] = \frac{1}{2^p\Gamma(p)L(t)^p}\int_0^1 (2-\theta)\theta^{p-1}(1-\theta)^{\frac{2a}{\sigma^2}-p-1}\exp\left(-\frac{xe^{-bt}\theta}{2L(t)}\right)d\theta,$$

from which we deduce (52). Now if $a > \sigma^2$, we have for $p = 1$

$$\mathbb{E}\left[\frac{1}{r_t^x}\right] \leq \frac{1}{2L(t)}\int_0^1 \exp\left(-\frac{xe^{-bt}\theta}{2L(t)}\right)d\theta \leq \frac{e^{bt}}{x}$$

and for $1 < p < \frac{2a}{\sigma^2} - 1$, $\mathbb{E}\left[\frac{1}{(r_t^x)^p}\right] \leq \frac{1}{2^p\Gamma(p)L(t)^p} = \frac{2^p|b|^p}{\sigma^{2p}\Gamma(p)(1-e^{-|b|t})^p}$ which gives (54), by noting that $(1-e^{-|b|t}) \geq |b|te^{-|b|t}$. $\square$

Lemma A.2. *If $a \geq \sigma^2/2$ and $b \geq 0$, there exists a constant C depending on a, b, σ and T, such that*

$$\sup_{t\in[0,T]}\mathbb{E}\exp\left(\frac{\nu^2\sigma^2}{8}\int_0^t\frac{ds}{r_s^x}\right) \leq C\left(1 + x^{-\frac{\nu}{2}}\right), \tag{55}$$

where $\nu = \frac{2a}{\sigma^2} - 1 \geq 0$.

Proof. For any $t \in [0,T]$, we set $H_t = \frac{2}{\sigma}\sqrt{r_t^x}$, so that $\mathbb{E}\exp(\frac{\nu^2\sigma^2}{8}\int_0^t \frac{ds}{r_s^x}) = \mathbb{E}\exp(\frac{\nu^2}{2}\int_0^t \frac{ds}{H_s^2})$. The process $(H_t, t \in [0,T])$ solves

$$dH_t = \left(\frac{2a}{\sigma^2} - \frac{1}{2}\right)\frac{dt}{H_t} - \frac{b}{2}H_t dt + dW_t, \;\; H_0 = \frac{2}{\sigma}\sqrt{x}.$$

For any $t \in [0,T]$, we set $B_t = H_t - H_0 - \int_0^t \left(\frac{2a}{\sigma^2} - \frac{1}{2}\right)\frac{ds}{H_s}$. Let $(\mathcal{Z}_t, t \in [0,T])$ defined by

$$\mathcal{Z}_t = \exp\left(-\int_0^t \frac{b}{2}H_s dB_s - \frac{b^2}{8}\int_0^t H_s^2 ds\right).$$

By the Girsanov Theorem, under the probability $\mathbb{Q}$ such that $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t}$, $(B_t, t \in [0,T])$ is a Brownian motion. Indeed $(H_t, t \in [0,T])$ solves

$$dH_t = \left(\frac{2a}{\sigma^2} - \frac{1}{2}\right)\frac{dt}{H_t} + dB_t, \; t \leq T, \;\; H_0 = \frac{2}{\sigma}\sqrt{x}$$

and under $\mathbb{Q}$, we note that (H_t) is a Bessel process with index $\nu = \frac{2a}{\sigma^2} - 1$. Moreover, by the integration by parts formula, $\int_0^t 2H_s dB_s = H_t^2 - H_0^2 - \frac{4a}{\sigma^2}t$ and

$$\mathcal{Z}_t = \exp\left(-\frac{b}{4}H_t^2 - \frac{b^2}{8}\int_0^t H_s^2 ds + \frac{b}{\sigma^2}x + t\frac{ba}{\sigma^2}\right) \leq \exp\left(\frac{b}{\sigma^2}x + T\frac{ba}{\sigma^2}\right).$$

Now, denoting by $\mathbb{E}^{\mathbb{Q}}$ the expectation relative to $\mathbb{Q}$,

$$\begin{aligned}\mathbb{E}\exp\left(\frac{\nu^2}{2}\int_0^t \frac{ds}{H_s^2}\right) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{\nu^2}{2}\int_0^t \frac{ds}{H_s^2}\right)\mathcal{Z}_t\right] \\ &\leq \exp\left(\frac{b}{\sigma^2}x + T\frac{ba}{\sigma^2}\right)\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{\nu^2}{2}\int_0^t \frac{ds}{H_s^2}\right)\right].\end{aligned}$$

Let $\mathbb{E}^{(\nu)}_{\frac{2}{\sigma}\sqrt{x}}$ denotes the expectation relative to $\mathbb{P}^{(\nu)}_{\frac{2}{\sigma}\sqrt{x}}$, the law on $C(\mathbb{R}^+, \mathbb{R}^+)$ of the Bessel process with index ν, starting at $\frac{2}{\sigma}\sqrt{x}$. The next step uses the following change of probability measure, for $\nu \geq 0$ (see Proposition 2.4 in [11]).

$$\left.\mathbb{P}^{(\nu)}_{\frac{2}{\sigma}\sqrt{x}}\right|_{\sigma(R_s, s\leq t)} = \left(\frac{\sigma R_t}{2\sqrt{x}}\right)^{\nu}\exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\left.\mathbb{P}^{(0)}_{\frac{2}{\sigma}\sqrt{x}}\right|_{\sigma(R_s, s\leq t)},$$

where $(R_t, t \geq 0)$ denotes the canonical process on $C(\mathbb{R}^+, \mathbb{R}^+)$. Then, we obtain that

$$\begin{aligned}\mathbb{E}\exp\left(\frac{\nu^2}{2}\int_0^t \frac{ds}{H_s^2}\right) &\leq \exp\left(\frac{b}{\sigma^2}x + T\frac{ba}{\sigma^2}\right)\mathbb{E}^{(\nu)}_{\frac{2}{\sigma}\sqrt{x}}\left[\exp\left(\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\right] \\ &\leq \exp\left(\frac{b}{\sigma^2}x + T\frac{ba}{\sigma^2}\right)\mathbb{E}^{(0)}_{\frac{2}{\sigma}\sqrt{x}}\left[\left(\frac{\sigma R_t}{2\sqrt{x}}\right)^{\nu}\right].\end{aligned}$$

It remains to compute $\mathbb{E}^{(0)}_{\frac{2}{\sigma}\sqrt{x}}\left[\left(\frac{\sigma R_t}{2\sqrt{x}}\right)^{\nu}\right]$. Let $(W_t^1, W_t^2, t \geq 0)$ be a two dimensional Brownian motion. Then

$$\mathbb{E}^{(0)}_{\frac{2}{\sigma}\sqrt{x}}\left[\left(\frac{\sigma R_t}{2\sqrt{x}}\right)^{\nu}\right] = \left(\frac{\sigma}{2\sqrt{x}}\right)^{\nu}\mathbb{E}\left[\left((W_t^1)^2 + (W_t^2 + \frac{2\sqrt{x}}{\sigma})^2\right)^{\frac{\nu}{2}}\right]$$

and an easy computation shows that $\mathbb{E}^{(0)}_{\frac{2}{\sigma}\sqrt{x}}\left[\left(\frac{\sigma R_t}{2\sqrt{x}}\right)^{\nu}\right] \leq C(T)\left(1 + x^{-\frac{\nu}{2}}\right)$. □

B Proofs of Propositions 3.4 and 4.4

Proof of Proposition 3.4. To simplify the presentation, we consider only the case when $k(x)$ and $h(x)$ are nil. For any $\epsilon > 0$ and $x > 0$, we define for all $t \in [0,\, T]$, the process $J_t^{x,\epsilon} = \frac{1}{\epsilon}(X_t^{x+\epsilon} - X_t^x)$, satisfying

$$J_t^{x,\epsilon} = 1 + \int_0^t \phi_s^\epsilon J_s^{x,\epsilon} ds + \int_0^t \psi_s^\epsilon J_s^{x,\epsilon} dW_s,$$

with $\phi_s^\epsilon = \int_0^1 b'\left(X_s^x + \theta\epsilon J_t^{x,\epsilon}\right) d\theta$ and $\psi_s^\epsilon = \int_0^1 \frac{\sigma d\theta}{2\sqrt{X_s^x + \epsilon\theta J_t^{x,\epsilon}}}$. Under (H3), the trajectories $(X_t^x, 0 \le t \le T)$ are strictly positive a.s. (see Remark 2.2). By Lemma A.1, $\int_0^t \psi_s^\epsilon dW_s$ is a martingale. Then $J_t^{x,\epsilon}$ is explicitly given by

$$J_t^{x,\epsilon} = \exp\left(\int_0^t \phi_s^\epsilon ds + \int_0^t \psi_s^\epsilon dW_s - \frac{1}{2}\int_0^t (\psi_s^\epsilon)^2 ds\right).$$

We remark that $\int_0^t \frac{\sigma}{2\sqrt{X_s^x}} dW_s = \frac{1}{2}\log\left(\frac{X_t^x}{x}\right) - \int_0^t \frac{1}{2}\frac{b(X_s^x)}{X_s^x} ds$ and

$$J_t^{x,\epsilon} \le C\sqrt{\frac{X_t^x}{x}} \exp\left(-\int_0^t \frac{1}{2}\frac{b(X_s^x)}{X_s^x} ds - \frac{1}{2}\int_0^t (\psi_s^\epsilon)^2 ds + \int_0^t \left(\psi_s^\epsilon - \frac{\sigma}{2\sqrt{X_s^x}}\right) dW_s\right).$$

We upper-bound the moments $\mathbb{E}(J_t^{x,\epsilon})^\alpha$, $\alpha > 0$. As $b(x) \ge b(0) - Kx$, for any p,

$$\begin{aligned}(J_t^{x,\epsilon})^\alpha \le & C\left(\frac{X_t^x}{x}\right)^{\frac{\alpha}{2}} \exp\left(-\int_0^t \frac{\alpha}{2}\frac{b(0)}{X_s^x} ds - \frac{\alpha}{2}\int_0^t (\psi_s^\epsilon)^2 ds + \int_0^t \frac{\alpha^2 p}{2}\left(\psi_s^\epsilon - \frac{\sigma}{2\sqrt{X_s^x}}\right)^2 ds\right) \\ & \times \exp\left(\int_0^t \alpha\left(\psi_s^\epsilon - \frac{\sigma}{2\sqrt{X_s^x}}\right) dW_s - \int_0^t \frac{\alpha^2 p}{2}\left(\psi_s^\epsilon - \frac{\sigma}{2\sqrt{X_s^x}}\right)^2 ds\right)\end{aligned}$$

and by the Hölder Inequality for $p > 1$, we have

$$\mathbb{E}(J_t^{x,\epsilon})^\alpha \le C\left\{\mathbb{E}\left[\left(\frac{X_t^x}{x}\right)^{\frac{\alpha p}{2(p-1)}} \exp\left(\frac{\alpha p}{2(p-1)}\left[-\int_0^t \frac{b(0)}{X_s^x} ds + \alpha p\int_0^t \frac{\sigma^2}{4X_s^x} ds\right]\right)\right]\right\}^{\frac{p-1}{p}}.$$

Then, for any $0 < \alpha < 4$, for any $p > 1$ such that $\alpha p \le 4$,

$$\mathbb{E}(J_t^{x,\epsilon})^\alpha \le \quad C\left\{\mathbb{E}\left[\left(\frac{X_t^x}{x}\right)^{\frac{\alpha p}{2(p-1)}}\right]\right\}^{\frac{p-1}{p}}. \tag{56}$$

The same computation shows that for the same couple (p, α) and for any $0 \le \beta \le \frac{p-1}{p}$,

$$\mathbb{E}\left(\frac{(J_t^{x,\epsilon})^\alpha}{(X_t^x)^{\frac{\alpha}{2}+\beta}}\right) \le \frac{C}{x^{\frac{\alpha}{2}}}\left\{\mathbb{E}\left[(X_t^x)^{-\frac{\beta p}{p-1}}\right]\right\}^{\frac{p-1}{p}}, \tag{57}$$

which is bounded according to Lemma 3.1. Hence, by (56) for $(\alpha, p) = (2, 2)$, there exists a positive constant C such that

$$\mathbb{E}(X_t^{x+\epsilon} - X_t^x)^2 \le C\epsilon^2,\ \ \forall\, t \in [0, T],$$

and $X_t^{x+\epsilon}$ tends X_t^x in probability. We consider now the process $(J_t^x, t \in [0, T])$ solution of (19). Applying the integration by parts formula in (20), we obtain that $J_t^x = \sqrt{\frac{X_t^x}{x}}\exp(\int_0^t (b'(X_s^x) - \frac{b(X_s^x)}{2X_s^x} + \frac{\sigma^2}{8}\frac{1}{X_s^x})ds)$, from which by using (H3), we have

$$J_t^x \le \sqrt{\frac{X_t^x}{x}} \exp\left(-\int_0^t (b(0) - \frac{\sigma^2}{4})\frac{ds}{2X_\theta^x}\right) \exp(KT) \le C\sqrt{\frac{X_t^x}{x}}. \tag{58}$$

Moreover,

$$J_t^x - J_t^{x,\epsilon} = \int_0^t b'(X_s^x)(J_s^x - J_s^{x,\epsilon})ds + \int_0^t \frac{\sigma}{2\sqrt{X_s^x}}(J_s^x - J_s^{x,\epsilon})dW_s$$
$$+ \int_0^t (b'(X_s^x) - \phi_s^\epsilon)J_s^{x,\epsilon}ds + \int_0^t (\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)J_s^{x,\epsilon}dW_s.$$

We study the convergence of $\mathbb{E}(J_t^x - J_t^{x,\epsilon})^2$ as ϵ tends to 0. We set $\mathcal{E}_t^x := |J_t^x - J_t^{x,\epsilon}|$. By the Itô's formula,

$$\mathbb{E}(\mathcal{E}_t^x)^2 = \mathbb{E}\int_0^t 2b'(X_s^x)(\mathcal{E}_s^x)^2ds + \mathbb{E}\int_0^t 2(b'(X_s^x) - \phi_s^\epsilon)J_s^{x,\epsilon}(J_s^x - J_s^{x,\epsilon})ds$$
$$+ \mathbb{E}\int_0^t \left(\frac{\sigma}{2\sqrt{X_s^x}}(J_s^x - J_s^{x,\epsilon}) + (\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)J_s^{x,\epsilon}\right)^2 ds.$$

We upper-bound the third term in the right-hand side of the expression above: as $\frac{\sigma}{2\sqrt{X_s^x}} \geq \psi_s^\epsilon$ and $(\frac{\sigma}{2\sqrt{X_s^x}} + \psi_s^\epsilon) \leq C/\sqrt{X_s^x}$,

$$\mathbb{E}\left((\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)J_s^{x,\epsilon}\right)^2 \leq C\mathbb{E}\left((\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)\frac{(J_s^{x,\epsilon})^2}{\sqrt{X_s^x}}\right).$$

An easy computation shows that $\sqrt{X_s^x}(\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon) \leq \sqrt{\epsilon}\frac{\sqrt{J_s^{x,\epsilon}}}{\sqrt{X_s^x}}$. Then,

$$\mathbb{E}\left((\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)J_s^{x,\epsilon}\right)^2 \leq C\sqrt{\epsilon}\mathbb{E}\left(\frac{(J_s^{x,\epsilon})^{\frac{5}{2}}}{(X_s^x)^{\frac{3}{2}}}\right) = C\sqrt{\epsilon}\mathbb{E}\left(\frac{(J_s^{x,\epsilon})^{\frac{5}{2}}}{(X_s^x)^{\frac{5}{4}+\frac{1}{4}}}\right) \leq C\sqrt{\epsilon},$$

where we have applied (57) with $(\alpha, p, \beta) = (\frac{5}{2}, \frac{8}{5}, \frac{1}{4} \leq \frac{p-1}{p} = \frac{3}{8})$. By using the same arguments with (58),

$$\mathbb{E}\left(\frac{\sigma}{2\sqrt{X_s^x}}(J_s^x - J_s^{x,\epsilon})(\frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon)J_s^{x,\epsilon}\right) \leq \mathbb{E}\left(\frac{\sigma}{2\sqrt{X_s^x}}(J_s^x + J_s^{x,\epsilon})\sqrt{\epsilon}\frac{\sqrt{J_s^{x,\epsilon}}}{X_s^x}J_s^{x,\epsilon}\right)$$
$$\leq C\sqrt{\epsilon}\left(\mathbb{E}\left(\frac{(J_s^{x,\epsilon})^{\frac{3}{2}}}{X_s^x}\right) + \mathbb{E}\left(\frac{(J_s^{x,\epsilon})^{\frac{5}{2}}}{(X_s^x)^{\frac{3}{2}}}\right)\right) \leq C\sqrt{\epsilon},$$

where we have applied (57) with $(\alpha, p, \beta) = (\frac{3}{2}, \frac{8}{3}, \frac{1}{4} \leq \frac{p-1}{p} = \frac{5}{3})$. An easy computation shows that $|b'(X_s^x) - \phi_s^\epsilon| \leq \epsilon J_s^{x,\epsilon}\|b''\|_\infty$. Coming back to the upper-bound of $\mathbb{E}(\mathcal{E}_t^x)^2$, we have

$$\mathbb{E}(\mathcal{E}_t^x)^2 \leq C\int_0^t \mathbb{E}(\mathcal{E}_s^x)^2ds + C\sqrt{\epsilon}t + \mathbb{E}\left(\int_0^t \frac{\sigma^2}{4X_s^x}(\mathcal{E}_s^x)^2ds\right). \tag{59}$$

To conclude on the convergence, as ϵ tends to 0, we use the stochastic time change technique introduced in [3] to analyze the strong rate of convergence. For any $\lambda > 0$, we define the stopping time τ_λ as

$$\tau_\lambda = \inf\{s \in [0,T],\ \gamma(s) \geq \lambda\} \text{ with } \gamma(t) = \int_0^t \frac{\sigma^2 ds}{4X_s^x} \text{ and } \inf\emptyset = T.$$

Then, by using the Lemma 3.1 with the Markov Inequality,

$$\mathbb{P}(\tau_\lambda < T) = \mathbb{P}(\gamma(T) \geq \lambda) \leq \exp(-\frac{\lambda}{2})\mathbb{E}\left(\exp\left(\int_0^T \frac{\sigma^2 ds}{8X_s^x}\right)\right) \leq C\exp(-\frac{\lambda}{2}).$$

Choosing $\lambda = -\log(\epsilon^r)$ for a given $r > 0$, we have that $\mathbb{P}(\tau_\lambda < T) \le C\epsilon^{\frac{r}{2}}$ and

$$\mathbb{E}(\mathcal{E}_T^x)^2 \le \mathbb{E}(\mathcal{E}_{\tau_\lambda}^x)^2 + C\epsilon^{\frac{r}{4}}.$$

With (59), we can easily check that for any bounded stopping time $\tau \le T$,

$$\mathbb{E}(\mathcal{E}_\tau^x)^2 \le \int_0^T \exp(C(T-s)) \left\{ \mathbb{E}\left(\int_0^\tau \frac{\sigma^2 ds}{4X_s^x} (\mathcal{E}_s^x)^2 \right) + C\sqrt{\epsilon} \right\}$$

and for τ_λ,

$$\mathbb{E}(\mathcal{E}_{\tau_\lambda}^x)^2 \le C_1 \mathbb{E}\left(\int_0^{\tau_\lambda} (\mathcal{E}_s^x)^2 d\gamma(s) \right) + C_0\sqrt{\epsilon},$$

for some positive constants C_0 and C_1, depending on T. After the change of time $u = \gamma(s)$, we can apply the Gronwall Lemma

$$\mathbb{E}(\mathcal{E}_{\tau_\lambda}^x)^2 \le C_1 \mathbb{E}\left(\int_0^\lambda (\mathcal{E}_{\tau_u}^x)^2 du \right) + C_0\sqrt{\epsilon} \le TC_0\sqrt{\epsilon}\exp(C_1\lambda).$$

With the choice $r = (4C_1)^{-1}$ and $\lambda = -\log(\epsilon^r)$, we get $\mathbb{E}(\mathcal{E}_{\tau_\lambda}^x)^2 \le TC_0\epsilon^{\frac{1}{4}}$. As T is arbitrary in the preceding reasoning, we conclude that $\mathbb{E}|J_t^x - J_t^{x,\epsilon}|$ tends to 0 with ϵ for all $t \in [0,T]$. Consider now

$$\begin{aligned}
&\frac{g(X_t^{x+\epsilon}) - g(X_t^x)}{\epsilon} - g'(X_t^x)J_t^x = J_t^{x,\epsilon}\int_0^1 g'(X_t^x + \epsilon\alpha J_t^{x,\epsilon})d\alpha - J_t^x g'(X_t^x) \\
&= (J_t^{x,\epsilon} - J_t^x)\int_0^1 g'(X_t^x + \epsilon\alpha J_t^{x,\epsilon})d\alpha + J_t^x \int_0^1 \left(g'(X_t^x + \epsilon\alpha J_t^{x,\epsilon}) - g'(X_t^x)\right) d\alpha \\
&:= A^\epsilon + B^\epsilon.
\end{aligned}$$

$\mathbb{E}A^\epsilon \le \|g'\|_\infty \mathbb{E}|J_t^x - J_t^{x,\epsilon}|$, which tends to zero with ϵ. B^ϵ is a uniformly integrable sequence. g' is a continuous function. By the Lebesgue Theorem, as $X_t^{x+\epsilon}$ tends X_t^x in probability, B^ϵ tends to 0 with ϵ. As a consequence, $\mathbb{E}(\frac{g(X_t^{x+\epsilon}) - g(X_t^x)}{\epsilon})$ tends to $\mathbb{E}[g'(X_t^x)J_t^x]$ when ϵ tends to 0. $\square$

Proof of Proposition 4.4. The proof is very similar to the proof of Proposition 3.4. Again, we consider only the case when $h(x)$ and $k(x)$ are nil. Let $J_t^{x,\epsilon} = \frac{1}{\epsilon}(X_t^{x+\epsilon} - X_t^x)$, given also by

$$J_t^{x,\epsilon} = \exp\left(\int_0^t \phi_s^\epsilon ds + \int_0^t \psi_s^\epsilon dW_s - \frac{1}{2}\int_0^t (\psi_s^\epsilon)^2 ds \right),$$

with $\phi_s^\epsilon = \int_0^1 b'(X_t^x + \theta\epsilon J_s^{x,\epsilon})d\theta$ and $\psi_t^\epsilon = \int_0^1 \frac{\alpha\sigma d\theta}{(X_t^x + \theta\epsilon J_s^{x,\epsilon})^{1-\alpha}}$. For any C^1 function $g(x)$ with bounded derivative, we have

$$\begin{aligned}
&\frac{g(X_t^{x+\epsilon}) - g(X_t^x)}{\epsilon} - g'(X_t^x)J_t^x = J_t^{x,\epsilon}\int_0^1 g'(X_t^x + \epsilon\theta J_t^{x,\epsilon})d\theta - J_t^x g'(X_t^x) \\
&= (J_t^{x,\epsilon} - J_t^x)\int_0^1 g'(X_t^x + \epsilon\theta J_t^{x,\epsilon})d\theta + J_t^x \int_0^1 \left(g'(X_t^x + \epsilon\theta J_t^{x,\epsilon}) - g'(X_t^x)\right) d\theta \\
&:= A^\epsilon + B^\epsilon.
\end{aligned}$$

$\mathbb{E}(J_t^{x,\epsilon})^\alpha \le \exp(\alpha\|b'\|_\infty t)\mathbb{E}\exp(\alpha\int_0^t \psi_s^\epsilon dW_s - \frac{\alpha}{2}\int_0^t (\psi_s^\epsilon)^2 ds)$ and, by using Lemma 4.2*(ii)*, one easily concludes that $\mathbb{E}(J_t^{x,\epsilon})^\alpha \le C$ and consequently that $X_t^{x+\epsilon}$ converges to X_t^x in $L^2(\Omega)$. Then, by applying the Lebesgue Theorem, $\mathbb{E}|B^\epsilon|$ tends to 0. Moreover, $\mathbb{E}|A^\epsilon| \le \|g'\|_\infty\sqrt{\mathbb{E}|J_t^{x,\epsilon} - J_t^x|^2}$. We can proceed as in the proof of the Proposition 3.4, to show that $\mathbb{E}|J_t^{x,\epsilon} - J_t^x|^2$ tends to 0, but now the moments $\mathbb{E}(J_t^{x,\epsilon})^\alpha, \alpha > 0$ are bounded and the Lemma 4.1 ensures that the $\mathbb{E}|X_t^x|^{-p}, p > 0$ are all bounded. $\square$

C End of the proof of Proposition 3.2

To compute $\frac{\partial^4 u}{\partial x^4}(t,x)$, we need first to avoid the appearance of $J_t^x(1)$ in the expression of $\frac{\partial^3 u}{\partial x^3}(t,x)$. We transform the expression of $\frac{\partial^3 u}{\partial x^3}(t,x)$ in (25), in order to obtain $\frac{\partial^3 u}{\partial x^3}(t,x)$ as a sum of terms of the form

$$\mathbb{E}\left(\exp\left(\int_0^{T-t}\beta(X_s^x(1))ds\right)\Gamma(X_{T-t}^x(1))J_{T-t}^x(1)\right)$$
$$+\int_0^{T-t}\mathbb{E}\left\{\exp\left(\int_0^s\beta(X_u^x(1))du\right)J_s^x(1)\Lambda(X_s^x(1))\right\}ds$$

for some functions $\beta(x)$, $\Gamma(x)$, $\Lambda(x)$. In this first step, to simplify the writing, we write X_s^x instead of $X_s^x(1)$. Two terms are not of this form in (25):

$$\mathrm{I}=2\mathbb{E}\left\{\exp\left(2\int_0^{T-t}b'(X_s^x)ds\right)f''(X_{T-t}^x)\int_0^{T-t}b''(X_s^x)J_s^x ds\right\}$$
$$\mathrm{II}=2\mathbb{E}\Big\{\int_0^{T-t}\exp\left(2\int_0^s b'(X_u^x)du\right)\frac{\partial u}{\partial x}(t+s,X_s^x)b''(X_s^x)\left(\int_0^s b''(X_u^x)J_u^x du\right)ds\Big\}.$$

The integration by parts formula gives immediately that

$$\mathrm{II}=2\mathbb{E}\left\{\int_0^{T-t}b''(X_s^x)J_s^x\left(\int_s^{T-t}\frac{\partial u}{\partial x}(t+u,X_u^x)\exp\left(2\int_0^u b'(X_\theta^x)d\theta\right)b''(X_u^x)du\right)ds\right\}.$$

By using again the Markov property and the time homogeneity of the process (X_t^x),

$$\mathbb{E}\left[\exp\left(2\int_s^{T-t}b'(X_\theta^x)d\theta\right)f''(X_{T-t}^x)\Big/\mathcal{F}_s\right]=\mathbb{E}\left[\exp\left(2\int_0^{T-t-s}b'(X_\theta^y)d\theta\right)f''(X_{T-t-s}^y)\right]\Big|_{y=X_s^x}$$

and, by using (24),

$$\begin{aligned}\mathrm{I}=&2\int_0^{T-t}\mathbb{E}\left\{b''(X_s^x)J_s^x\exp\left(2\int_0^s b'(X_\theta^x)d\theta\right)\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x)\right\}ds\\&-2\int_0^{T-t}\mathbb{E}\left\{b''(X_s^x)J_s^x\exp\left(2\int_0^s b'(X_\theta^x)d\theta\right)\right.\\&\left.\times\left(\int_0^{T-t-s}\mathbb{E}\left[\frac{\partial u}{\partial x}(t+s+u,X_u^y)\exp\left(2\int_0^u b'(X_\theta^y)d\theta\right)b''(X_u^y)\right]\Big|_{y=X_s^x}du\right)\right\}ds.\end{aligned}$$

Conversely,

$$\begin{aligned}&\int_0^{T-t-s}\mathbb{E}\left[\frac{\partial u}{\partial x}(t+s+u,X_u^y)\exp\left(2\int_0^u b'(X_\theta^y)d\theta\right)b''(X_u^y)\right]\Big|_{y=X_s^x}du\\&=\int_s^{T-t}\mathbb{E}\left[\frac{\partial u}{\partial x}(t+u,X_u^x)\exp\left(2\int_s^u b'(X_\theta^x)d\theta\right)b''(X_u^x)\Big/\mathcal{F}_s\right]du\end{aligned}$$

and then

$$\begin{aligned}\mathrm{I}=&2\int_0^{T-t}\mathbb{E}\left\{b''(X_s^x)J_s^x\exp\left(2\int_0^s b'(X_\theta^x)d\theta\right)\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x)\right\}ds\\&-2\mathbb{E}\left\{\int_0^{T-t}b''(X_s^x)J_s^x\left(\int_s^{T-t}\frac{\partial u}{\partial x}(t+u,X_u^x)\exp\left(2\int_0^u b'(X_\theta^x)d\theta\right)b''(X_u^x)du\right)ds\right\}.\end{aligned}$$

Finally, replacing I and II in (25), we get

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(t,x) =& \mathbb{E}\left\{\exp\left(2\int_0^{T-t} b'(X_s^x)ds\right) f^{(3)}(X_{T-t}^x)J_{T-t}^x\right\} \\ &+ \int_0^{T-t}\mathbb{E}\left\{\exp\left(2\int_0^s b'(X_u^x)du\right) J_s^x \right. \\ &\left. \times\left(3\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x)b''(X_s^x) + \frac{\partial u}{\partial x}(t+s,X_s^x)b^{(3)}(X_s^x)\right)\right\} ds.\end{aligned}$$

To eliminate J_t^x, we introduce the probability $\mathbb{Q}^{3/2}$ such that $\left.\frac{d\mathbb{Q}^{3/2}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t^{(1,\frac{3}{2})}}$. Then

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(t,x) =& \mathbb{E}^{3/2}\left\{\exp\left(2\int_0^{T-t} b'(X_s^x)ds\right) f^{(3)}(X_{T-t}^x)\mathcal{Z}_{T-t}^{(1,\frac{3}{2})}J_{T-t}^x\right\} \\ &+ \int_0^{T-t}\mathbb{E}^{3/2}\left\{\exp\left(2\int_0^s b'(X_u^x)du\right) \mathcal{Z}_s^{(1,\frac{3}{2})}J_s^x \right. \\ &\left. \times\left(3\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x)b''(X_s^x) + \frac{\partial u}{\partial x}(t+s,X_s^x)b^{(3)}(X_s^x)\right)\right\} ds.\end{aligned}$$

Again, we note that $\mathcal{Z}_t^{(1,\frac{3}{2})}J_t^x = \exp\left(\int_0^t b'(X_u^x)du\right)$ and

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(t,x) =& \mathbb{E}^{3/2}\left\{\exp\left(3\int_0^{T-t} b'(X_s^x)ds\right) f^{(3)}(X_{T-t}^x)\right\} \\ &+ \int_0^{T-t}\mathbb{E}^{3/2}\left\{\exp\left(3\int_0^s b'(X_u^x)du\right) \right. \\ &\left. \times\left(3\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x)b''(X_s^x) + \frac{\partial u}{\partial x}(t+s,X_s^x)b^{(3)}(X_s^x)\right)\right\} ds\end{aligned}$$

where we write $X_\cdot^x$ instead of $X^x(1)$.. Finally, as $\mathcal{L}^{\mathbb{Q}^{3/2}}(X^x(1)) = \mathcal{L}^{\mathbb{P}}(X^x(\frac{3}{2}))$, we obtain the following expression for $\frac{\partial^3 u}{\partial x^3}(t,x)$:

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(t,x) =& \mathbb{E}\left\{\exp\left(3\int_0^{T-t} b'(X_s^x(\frac{3}{2}))ds\right) f^{(3)}(X_{T-t}^x(\frac{3}{2}))\right\} \\ &+ \int_0^{T-t}\mathbb{E}\left\{\exp\left(3\int_0^s b'(X_u^x(\frac{3}{2})du\right)\left(3\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x(\frac{3}{2}))b''(X_s^x(\frac{3}{2})) \right.\right. \\ &\left.\left. + \frac{\partial u}{\partial x}(t+s,X_s^x(\frac{3}{2}))b^{(3)}(X_s^x(\frac{3}{2}))\right)\right\} ds.\end{aligned}$$

$J_s^x(\frac{3}{2})$ exists and is given by (20). By the Proposition 3.4, $\frac{\partial^3 u}{\partial x^3}(t,x)$ is continuously differentiable and

$$\begin{aligned}\frac{\partial^4 u}{\partial x^4}(t,x) =&\mathbb{E}\left\{\exp\left(3\int_0^{T-t} b'(X_s^x(\frac{3}{2}))ds\right)\right.\\ &\left.\times\left[3f^{(3)}(X_{T-t}^x(\frac{3}{2}))\int_0^{T-t} b''(X_s^x(\frac{3}{2}))J_s^x(\frac{3}{2})ds + f^{(3)}(X_{T-t}^x(\frac{3}{2}))J_{T-t}^x\frac{3}{2})\right]\right\}\\ &+\int_0^{T-t}\mathbb{E}\left\{\exp\left(3\int_0^s b'(X_u^x(\frac{3}{2}))du\right)\left(3\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x(\frac{3}{2}))b''(X_s^x(\frac{3}{2}))\right.\right.\\ &\left.\left.+\frac{\partial u}{\partial x}(t+s,X_s^x(\frac{3}{2}))b^{(3)}(X_s^x(\frac{3}{2}))\right)\int_0^s 3b''(X_u^x(\frac{3}{2}))J_u^x(\frac{3}{2})du\right\}ds\\ &+\int_0^{T-t}\mathbb{E}\left\{\exp\left(3\int_0^s b'(X_u^x(\frac{3}{2}))du\right)J_s^x(\frac{3}{2})\left(3\frac{\partial^3 u}{\partial x^3}(t+s,X_s^x(\frac{3}{2}))b''(X_s^x(\frac{3}{2}))\right.\right.\\ &+4\frac{\partial^2 u}{\partial x^2}(t+s,X_s^x(\frac{3}{2}))b^{(3)}(X_s^x(\frac{3}{2}))\\ &\left.\left.+\frac{\partial u}{\partial x}(t+s,X_s^x(\frac{3}{2}))b^{(4)}(X_s^x(\frac{3}{2}))\right)\right\}ds,\end{aligned}$$

from which we can conclude on (16).

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